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SUPERMODULES, SUPERCOMODULES AND TWISTINGS OF FINITE-DIMENSIONAL HOPF SUPERALGEBRAS

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ABSTRACT. The purpose of this paper is to describe supermodule, supercomodule and supermodule superalgebra structures of finite dimensional Hopf superalgebras. In particular, it aims to enhance the work on classification of finite-dimensional Hopf superalgebras considered by Aissaoui and Makhlouf. Moreover, we discuss twist deformations and 2-cocycle deformations, extending to $\mathbb{Z}_2$-grading case Drinfeld's theory and the work by Giaquinto and Zhang. Furthermore, applications are provided.

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INTRODUCTION

Modules or supermodules over superalgebras have applications in a wide range of mathematical and physical fields. They provide a means to study the notion of supersymmetry in theoretical physics and can be viewed as a generalization of vector superspaces over a field. In the language of category theory, the class of all supermodules over a superalgebra forms a category with supermodule homomorphisms as the morphisms. This category is a symmetric monoidal closed category under the super tensor product whose internal Hom functor is given by Hom. We refer for basics to [7, 11, 15, 16, 17, 19].

Superbialgebras and Hopf superalgebras are not extensively studied, a classification of finite dimensional Hopf superalgebras up to dimension 4 was established by Aissaoui and Makhlouf [1]. In dimension 4, it is shown that there is five non pairwise isomorphic Hopf superalgebras, while for Hopf algebra we do have only four. As a general result we mention the study of pointed Hopf superalgebras by Andruskiewitsch, Angiono and Yamane in [2]. There is moreover some results in connection with quantum supergroups, see for example [6].

Twisting elements were introduced by Drinfeld [9] on quasi-Hopf algebras, in order to twist the coproduct without changing its product. They have become an important tool in the classification of finite dimensional Hopf algebras. Many works are focused on the construction of new algebras by using twisting method. Giaquinto and Zhang [12] studied a twist applied to algebra structure, they obtain a new algebra by twisting the multiplication of a given algebra. Dually, 2-cocycle deformations is a way to modify an algebraic structure by introducing a new operation that satisfies a 2-cocycle condition. This concept is widely used in different areas of mathematics to study deformations of algebraic structures and their properties. A 2-cocycle deformation of a superbialgebra is one where the coproduct is unchanged, while a new multiplication is defined via a formula which only depends on the initial multiplication and on a 2-cocycle σ of superbialgebra. The relationship between Drinfeld twists and 2-cocycles is that every Drinfeld twist corresponds to a 2-cocycle, and every 2-cocycle corresponds to a Drinfeld twist. This duality is known as the Drinfeld twist theorem, and it plays an important role in many areas of mathematics and physics, including quantum field theory, quantum groups and noncommutative geometry.

In this paper, we aim to study supermodule and supercomodule structures of finite dimensional Hopf superalgebras. In particular, we determine low dimensional supermodule, supercomodule and supermodule superalgebra structures for all Hopf superalgebras obtained in the classification established in [1]. Moreover, we study 2-cocycle deformations and twist deformations. In particular, we provide appropriate definitions and generalize the theorem given by Giaquinto-Zhang [12] in the $\mathbb{Z}_2$-grading case. Furthermore, we show some applications.

The paper is organized as follows. In Section 1, we review definitions and properties of superbialgebras and Hopf superalgebras, also we recall the classification of 4-dimensional Hopf superalgebras established in [1]. In Section 2, we introduce and discuss properties of supermodule, supercomodule and supermodule superalgebra structures. We describe all the low-dimensional ones for all Hopf superalgebras obtained in the classification given by [1]. In Section 3, we study deformations based on Drinfeld twisting. We generalize to $\mathbb{Z}_2$-grading case the construction of a new multiplication along a twisting on a B-supermodule superalgebra, where B is a superalgebra, see [12] for the nongraded case. In the last section, we deal with a dual version, considering 2-cocycles to construct new multiplications. These sections includes applications involving examples from previous classifications.

Throughout this paper, we work over $\mathbb{K}$, an algebraically closed field of characteristic zero. We will denote the supermodule and supermodule superalgebra structure by $\triangleright$ and supercomodule structure by ρ. For simplicity, we will just use the term supermodule (resp. supercomodule and supermodule superalgebra) as a synonym of left supermodule (resp. left supercomodule and left supermodule superalgebra). Notice also that all the morphisms considered in this paper are even.

1. Preliminaries

In this section, we provide some basics, definitions and properties of superbialgebras and Hopf superalgebras [3, 13, 18]. Moreover, we recall the classification of 4-dimensional Hopf superalgebras given in [1].

1.1. **Definitions and Properties.** A superspace A is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space $A = A_0 \oplus A_1$ such as A_0 is the even part and A_1 is the odd part. The elements of A_0 (resp. of A_1) are called *even homogeneous* (resp. *odd homogeneous*). We set $|a| = deg(a) = 0$ if $a \in A_0$ and $|a| = deg(a) = 1$ if $a \in A_1$. Notice that in the sequel all superspace morphisms are considered to be even.

Definition 1.1. *A superalgebra is a triple* (A, μ, η) *consisting of a superspace* A*, a product* $\mu : A \otimes A \to A$ *and a unit* $\eta : \mathbb{K} \to A$*, which are two morphisms satisfying the two following properties:*

1. *the product* μ *is associative i.e.* $\mu \circ (\mu \otimes id_A) = \mu \circ (id_A \otimes \mu)$*,*
2. *the unit axiom i.e.* $\mu \circ \eta \otimes id_A = \mu \circ id_A \otimes \eta$*.*

The unit map η is sometimes denoted by 1, when there is no confusion.

We say that a superalgebra A is commutative if the product satisfies $\mu \circ \tau = \mu$, where τ is a flip map, that is $\mu(a \otimes b) = (-1)^{|a||b|}\mu(b \otimes a)$ for all $a, b \in A$.

Let (A, μ_A, η_A) and (B, μ_B, η_B) be two superalgebras. A superalgebra morphism is a linear map $f : A \to B$ such that $f \circ \mu_A = \mu_B \circ f \otimes f$ and $f \otimes \eta_A = \eta_B$.

Definition 1.2. *A supercoalgebra is a triple* (C, Δ, ϵ) *consisting of a superspace* C*, a coproduct* $\Delta : C \to C \otimes C$ *and a counit* $\eta : \mathbb{K} \to A$ *which are two morphisms satisfying the two following properties:*

1. *the coproduct* Δ *is coassociative i.e.* $(\Delta \otimes id_C) \circ \Delta = (id_C \otimes \Delta) \circ \Delta$*,*
2. *the counit axiom i.e.* $(\epsilon \otimes id_C) \circ \Delta = (id_C \otimes \epsilon) \circ \Delta$*.*

We use Sweedleer's notation for the coproduct, we set for all $x \in C$

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}, \quad (\Delta \otimes id) \circ \Delta(x) = (id \otimes \Delta) \circ \Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} \otimes x_{(3)}.$$

We say that a supercoalgebra C is cocommutative if the coproduct satisfies $\Delta \circ \tau = \Delta$, where τ is a flip map, that is $\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} (-1)^{|x_{(1)}||x_{(2)}|} x_{(2)} \otimes x_{(1)}$ for all $x \in C$.

Let (A, μ_A, η_A), (B, μ_B, η_B) be two supercoalgebras. A supercoalgebra morphism is a linear map $f : A \to B$ such that $f \otimes f \circ \Delta_A = \Delta_B \circ f$ and $\epsilon_B \circ f = \epsilon_A$.

Definition 1.3. *A superbialgebra is a tuple* $(A, \mu, \eta, \Delta, \epsilon)$*, where* (A, μ, η) *is a superalgebra and* (A, Δ, ϵ) *is a supercoalgebra such that one of the following equivalent conditions is satisfied:*

(1) $\Delta : A \to A \otimes A$ *and* $\epsilon : A \to \mathrm{K}$ *are superalgebra morphisms.*
(2) $\mu : A \otimes A \to A$ *and* $\eta : \mathbb{K} \to A$ *are supercoalgebra morphisms.*

In other words, Δ (resp. ϵ) satisfies the following compatibility conditions

$$\Delta \circ \mu = (\mu \otimes \mu) \circ (Id_A \otimes \tau \otimes Id_A) \circ (\Delta \otimes \Delta), \quad \Delta \circ \eta = \eta \otimes \eta,$$

$$(\text{resp.} \quad \epsilon \circ \mu = \mu_{\mathbb{K}} \circ (\epsilon \otimes \epsilon), \quad \epsilon \circ \eta = id_{\mathbb{K}}),$$

where $\mu_{\mathbb{K}}$ is the multiplication of $\mathbb{K}$.
The second condition says that the unit element e_1^0 is a grouplike element, that is $\Delta(e_1^0) = e_1^0 \otimes e_1^0$.

Definition 1.4. *The convolution superalgebra of a supercoalgebra* (C, Δ, ε) *and a superalgebra* (A, μ, η) *over* $\mathbb{K}$ *is the superspace* $Hom(C, A)$ *with product defined by*

$$f * g = \mu \circ (f \otimes g) \circ \Delta.$$

for all $f, g \in Hom(C, A)$ *and identity* $\eta \circ \varepsilon$*.*

Definition 1.5. *A Hopf superalgebra is a superbialgebra $(A, \mu, \eta, \Delta, \epsilon)$ admitting an antipode, that is a morphism $S : A \to A$ which satisfies the following condition :*

$$\mu \circ (S \otimes id_A) \circ \Delta = \mu \circ (id_A \otimes S) \circ \Delta = \eta \circ \epsilon.$$

A Hopf superalgebra is denoted by a tuple $H = (A, \mu, \eta, \Delta, \epsilon, S)$.

Let $(A, \mu, \eta, \Delta, \epsilon, S)$ be a Hopf superalgebra and S its antipode, then we have the following properties:
(1) $S \circ \mu = \mu \circ \tau \circ (S \otimes S)$, (2) $S \circ \eta = \eta$, (3) $\epsilon \circ S = \epsilon$, (4) $\tau \circ (S \otimes S) \circ \Delta = \Delta \circ S$, (5) If A is commutative or cocommutative then $S \circ S = I$, where $I : A \longrightarrow A$ is the identity map.

Example 1.6 ([13])**.** *The universal enveloping algebra $U(\mathfrak{g})$ of a Lie superalgebra $\mathfrak{g}$ has a structure of cocommutative Hopf superalgebra, with the coproduct defined by $\Delta(x) = x \otimes 1 + 1 \otimes x$, counit by $\varepsilon(1) = 1$, $\varepsilon(x) = 0$ and antipode by $S(x) = -x$ for all $x \in \mathfrak{g}$. We extend these definitions to all element of $U(\mathfrak{g})$ by linearity and compatibility condition. Note that $\tau \circ \Delta = \Delta$ and $S^2 = I$.*

1.2. **Classification of low dimensional Hopf superalgebras.** In this section, we recall the classification of n-dimensional Hopf superalgebras, with $n = 2, 3, 4$, given in [1].
First, notice that connected n-dimensional Hopf superalgebras, that is Hopf superalgebras such that $\dim A_0 = 1$, exist only when $n = 1$ or 2.

Theorem 1.7. *Every non-trivial 2-dimensional Hopf superalgebra is isomorphic to the 2-dimensional Hopf superalgebra $\mathcal{H} = \mathbb{K}[x]/(x^2)$ with $\deg(x) = 1$ and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

Theorem 1.8. *There is no non-trivial 3-dimensional Hopf superalgebra.*

Theorem 1.9. *Every non-trivial 4-dimensional Hopf superalgebra, where $\dim A_0 = 3$, is isomorphic to the 4-dimensional Hopf superalgebra $\mathcal{H}_0 = \mathbb{K}[x, y]/(x^2 + y^2 - 1, xy)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that*

$$\Delta(x) = x \otimes x - \alpha y \otimes y, \quad \epsilon(x) = 1, \quad S(x) = x,$$
$$\Delta(y) = x \otimes y + y \otimes x, \quad \epsilon(y) = 0, \quad S(y) = \alpha y,$$

where α is a primitive 4^{th} root of unity.

Every non-trivial 4-dimensional Hopf superalgebra, where $\dim A_0 = 2$, is isomorphic to one of the following pairwise non-isomorphic Hopf superalgebras:

(1) $\mathcal{H}_1 = \mathbb{K}[x, y]/(x^2 - x, y^2)$ *with* $\deg(x) = 0$, $\deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes 1,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = x, \ S(y) = -y.$$

(2) $\mathcal{H}_2 = \mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, xy + yx - y)$ *with* $\deg(x) = 0$, $\deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes 1 - 2x \otimes y - 2y \otimes x,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = x, \; S(y) = y.$$

(3) $\mathcal{H}_3 = \mathbb{K}\langle x, y\rangle/(x^2, y^2, xy + yx)$ *with* $\deg(x) = \deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \Delta(y) = 1 \otimes y + y \otimes 1,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = -x, \; S(y) = -y.$$

(4) $\mathcal{H}_4 = \mathbb{K}[x, y]/(x^2 - x, y^2)$ *with* $\deg(x) = 0$, $\deg(y) = 1$ *and such that*

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes 1 - 2x \otimes y,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = x, \; S(y) = -y + 2xy,$$

where $\mathbb{K}\langle x, y\rangle$ *stands for noncommutative polynomials.*

1.3. **Supermodules and Supercomodules.** In this section, we define supermodule, supercomodule and supermodule superalgebra structures over Hopf superalgebras. Then we construct all low dimensional supermodules, supercomodules and supermodule superalgebras for Hopf superalgebras obtained in the classification established in [1] and recalled in the previous section.

We will use just the term supermodule (resp. supercomodule and supermodule superalgebra) as a synonym of left supermodule (resp. left supercomodule and left supermodule superalgebra).

Definition 1.10. *A supermodule over a superalgebra* (A, μ_A, η_A) *is a pair* $(V, \triangleright)$ *consisting of a superspace* V *over* $\mathbb{K}$ *and a morphism* $\triangleright : A \otimes V \longrightarrow V$ *such that the following identities hold for all* $a, b \in A$ *and* $v \in V$

$$(ab) \triangleright v = a \triangleright (b \triangleright v). \tag{1.1}$$
$$1_A \triangleright v = v. \tag{1.2}$$

Remark 1.11. (1) *A supermodule over a* $\mathbb{K}$*-superalgebra* A *can be defined equivalently as a pair* (V, φ) *of a* $\mathbb{K}$*-superspace* V *and a superalgebra homomorphism* $\varphi : A \longrightarrow End_{\mathbb{K}}(V)$.

(2) *A supermodule homomorphism is a morphism* $f : V \longrightarrow W$ *such that*

$$f \circ \triangleright_V = \triangleright_W \circ (id \otimes f). \tag{1.3}$$

(3) *If* $(V, \triangleright_V)$ *and* $(W, \triangleright_W)$ *are supermodules over* $\mathbb{K}$*-superalgebras* A *and* B*, then*

$$\triangleright : (A \otimes B) \otimes (V \otimes W) \longrightarrow V \otimes W, \; (a \otimes b) \triangleright (v \otimes w) = (-1)^{|b||v|}(a \triangleright_V v) \otimes (b \triangleright_W w)$$

defines an $A \otimes B$*-supermodule structure on* $V \otimes W$.

Definition 1.12. *A supercomodule over a supercoalgebra* $(C, \Delta_C, \epsilon_C)$ *is a pair* (V, ρ) *consisting of a superspace* V *over* $\mathbb{K}$ *and a morphism* $\rho : V \longrightarrow C \otimes V$ *such that the following identities hold for all* $v \in V$

$$(id \otimes \rho) \circ \rho(v) = (\Delta_C \otimes id) \circ \rho(v). \tag{1.4}$$
$$(\varepsilon_C \otimes id) \circ \rho(v) = 1_{\mathbb{K}} \otimes v. \tag{1.5}$$

Remark 1.13. *We use the Sweedler's notation and set, for a supercomodule map $\rho : V \longrightarrow C \otimes V$, $\rho(v) = \sum_{(v)} v_{(1)} \otimes v_{(2)}$, where $v_{(1)}$ is interpreted as an element of C, $v_{(2)}$ as an element of V and $\sum_{(v)}$ as a finite sum over elements of $C \otimes V$. By definition of a supercomodule, one then has*

$$\begin{aligned}(id \otimes \rho) \circ \rho(v) &= \sum_{(v)} v_{(1)} \otimes v_{(2)(1)} \otimes v_{(2)(2)} = \sum_{(v)} v_{(1)} \otimes v_{(2)} \otimes v_{(3)} \\ &= \sum_{(v)} v_{(1)(1)} \otimes v_{(1)(2)} \otimes v_{(2)} = (\Delta_C \otimes id) \circ \rho(v). \\ \textstyle\sum_{(v)} \epsilon_C(v_{(1)}) \otimes v_{(2)} &= 1_{\mathbb{K}} \otimes v.\end{aligned}$$

A supercomodule homomorphism is a morphism $f : V \longrightarrow W$ such that

$$(id \otimes f) \circ \rho_V = \rho_W \circ f. \tag{1.6}$$

Definition 1.14. *Let H be a superbialgebra or a Hopf superalgebra. A supermodule superalgebra over H is a superalgebra (A, μ_A, η_A) together with an H-supermodule structure $\rhd : H \otimes A \longrightarrow A$ such that the following identities hold for all $h \in H$ and $a, a' \in A$*

$$h \rhd (aa') = \sum_{(h)} (-1)^{|h_{(2)}||a|} (h_{(1)} \rhd a)(h_{(2)} \rhd a'), \tag{1.7}$$

$$h \rhd 1_A = \epsilon(h) 1_A. \tag{1.8}$$

Remark 1.15. (1) *Every Hopf superalgebra H is a supermodule superalgebra over itself with the adjoint action*

$$\rhd_{ad} : H \otimes H \longrightarrow H, h \otimes k \mapsto h \rhd_{ad} k = \sum_{(h)} (-1)^{|h_{(2)}||k|} h_{(1)} \cdot k \cdot S(h_{(2)}).$$

(2) *Let H be a superbialgebra over $\mathbb{K}$, A be a superalgebra over $\mathbb{K}$, and suppose that $f \in Hom(H, A)$ is a superalgebra morphism with convolution inverse. Then A is an H-supermodule superalgebra where*

$$h \rhd a = \sum_{(h)} (-1)^{|h_{(2)}||a|} f(h_{(1)}) a f^{-1}(h_{(2)}), \qquad \forall h \in H, \forall a \in A.$$

2. Towards Classification of supermodules, supercomodules and supermodule superalgebras

We aim in the section to determine the structures of supermodule (resp. supercomodule, supermodule superalgebra) over a superalgebra (resp. supercoalgebra, Hopf superalgebra). A supermodule (resp. supercomodule, supermodule superalgebra) is identified by its structure constants with respect to a fixed basis. It turns out that the axioms of a supermodule structure (resp. supercomodule structure, supermodule superalgebra structure) can be translated into a system of polynomial equations. The resolution of the polynomial equations system lead to a description of these structures.

Let (A, μ_A, η_A) be a superalgebra of dimension $n = n_0 + n_1$, where n_0 is the even part dimension and n_1 is the odd part dimension. Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 the odd part dimension.

We set $\{e_i^s\}_{s=0,1;i=1,\dots,n_s}$ be a basis of A and $\{v_p^q\}_{q=0,1;p=1,\dots,m_q}$ be a basis of

V. We identify the multiplication μ_A and the action $\triangleright$ with their structure constants $C^k_{(i,s)(j,t)}$ and $M^k_{(i,s)(p,q)}$ respectively, where

$$\mu_A(e^s_i \otimes e^t_j) = e^s_i \cdot e^t_j = \sum_{h=1}^{n_r} C^h_{(i,s)(j,t)} e^r_h \quad \text{with } r = (s+t)\text{mod}[2],$$

$$e^s_i \triangleright v^q_p = \sum_{k=1}^{m_z} M^k_{(i,s)(p,q)} v^z_k \quad \text{with } z = (s+q)\text{mod}[2].$$

The collection $\{M^k_{(i,s)(p,q)}\}$ with $s, q = 0, 1; i = 1, ..., n_s, p = 1, ..., m_q$ represent a supermodule over A if the action $\triangleright$ satisfy Conditions ((1.1),(1.2)) which translate to the following polynomial equations:

Condition (1.1) expresses as

$$\sum_{h=1}^{n_r} M^l_{(h,r)(p,q)} C^h_{(i,s)(j,t)} - \sum_{k=1}^{m_{z'}} M^l_{(i,s)(k,z')} M^k_{(j,t)(p,q)} = 0,$$

where $s, t, q = \{0,1\}^3$, $i = 1, \dots, n_s$, $j = 1, \dots, n_t$, $p = 1, \dots, m_q$, $l = 1, \dots, m_{(s+t+q)\text{mod}[2]}$, $z' = (t+q)\text{mod}[2]$.

Condition (1.2) expresses as

$$M^p_{(1,0)(p,q)} = 1, \quad M^k_{(1,0)(p,q)} = 0 \quad \text{when } k \neq p,$$

where $q = \{0,1\}$, $p = 1, \dots, m_q$, $k = 1, \dots, m_q$.

Let $(C, \Delta_C, \varepsilon_C)$ be a supercoalgebra of dimension $n = n_0 + n_1$. Let $\{e^l_i\}_{l=0,1;i=1,\dots,n_l}$ be a basis of C and $\{v^q_p\}_{q=0,1;p=1,\dots,m_q}$ be a basis of V. We identify the comultiplication Δ_C, counit ε_C and the coaction ρ with their structure constants $D^{(j,s)(k)}_{(i,l)}$, ξ^0_i and $R^{(u,w)(h)}_{(p,q)}$ respectively, where

$$\Delta(e^l_i) = \sum_{s=0}^{1} \sum_{j=1}^{n_s} \sum_{k=1}^{n_t} D^{(j,s)(k)}_{(i,l)} e^s_j \otimes e^t_k \quad \text{with } t = (l+s)\text{mod}[2].$$

$$\varepsilon(e^l_i) = \begin{cases} \xi^0_i & \text{if } l = 0 \text{ and } i \neq 1, \\ 0 & \text{if } l = 1, \\ 1 & \text{if } l = 0 \text{ and } i = 1. \end{cases}$$

$$\rho(v^q_p) = \sum_{w=0}^{1} \sum_{u=1}^{n_w} \sum_{h=1}^{m_r} R^{(u,w)(h)}_{(p,q)} e^w_u \otimes v^r_h \quad \text{with } r = (q+w)\text{mod}[2].$$

The collection $\{R^{(u,w)(h)}_{(p,q)}\}$ with $q, w = 0, 1; u = 1, ..., n_w, p = 1, ..., m_q$ represent a supercomodule if the coaction ρ satisfy Conditions ((1.4),(1.5)), which translates to the following polynomial equations:
Condition (1.4) expresses as

$$\sum_{z=1}^{m_q} R^{(b,s')(t')}_{(z,q)} R^{(a,0)(z)}_{(p,q)} - \sum_{z'=1}^{n_{s'}} D^{(a,0)(b)}_{(z',s')} R^{(z',s')(t')}_{(p,q)} = 0,$$

where $q, s' = \{0,1\}^2$, $p = 1, ..., m_q$, $a = 1, ..., n_{(s'+1)\text{mod}[2]}$, $b = 1, ..., n_{s'}$, $t' = 1, ..., m_{(q+s')\text{mod}[2]}$.

$$\sum_{z=1}^{m_{(q+1)\text{mod}[2]}} R_{(z,(q+1)\text{mod}[2])}^{(b,s')(t')} R_{(p,q)}^{(a,1)(z)} - \sum_{z'=1}^{n_{(s'+1)\text{mod}[2]}} D_{(z_1,(s'+1)\text{mod}[2])}^{(a,1)(b)} R_{(p,q)}^{(z',(s'+1)\text{mod}[2])(t')} = 0,$$

where $q, s' = \{0,1\}^2$, $p = 1, ..., m_q$, $a = 1, ..., n_1$, $b = 1, ..., n_{s'}$, $t' = 1, ..., m_{(q+s'+1)\text{mod}[2]}$.

Condition (1.5) expresses as

$$R_{(p,q)}^{(1,0)(p)} + \sum_{j=2}^{n_0} R_{(p,q)}^{(j,0)(p)} \xi_j^0 = 1,$$

$$R_{(p,q)}^{(1,0)(h)} + \sum_{j=2}^{n_0} R_{(p,q)}^{(j,0)(h)} \xi_j^0 = 0, \quad \text{when } h \neq p,$$

where $q = \{0,1\}$, $p = 1, ..., m_q$, $h = 1, ..., m_q$.

Let H be a Hopf superalgebra of dimension $n = n_0 + n_1$. Let (A, μ_A, η_A) be a superalgebra of dimension $m = m_0 + m_1$. We set $\{e_i^s\}_{s=0,1;i=1,...,n_s}$ to be a basis of H and $\{v_p^q\}_{q=0,1;p=1,...,m_q}$ to be a basis of A. We identify the comultiplication and counit of H, multiplication of A and the action $\triangleright$ with their structure constants $D_{(i,l)}^{(j,s)(k)}$, ξ_i^0, $C_{(p,q)(u,w)}^{k'}$ and $M_{(i,s)(p,q)}^h$ respectively.

Condition (1.7) expresses as

$$\sum_{h''=1}^{m_{z''}} \sum_{h'=1}^{m_{z'}} \sum_{s=0}^{1} \sum_{j=1}^{n_s} \sum_{k=1}^{n_t} (-1)^{(qt)} D_{(i,l)}^{(j,s)(k)} M_{(j,s)(p,q)}^{h'} M_{(k,t)(u,w)}^{h''} C_{(h',z')(h'',z'')}^{a} - \sum_{k'=1}^{m_r} C_{(p,q)(u,w)}^{k'} M_{(i,l)(k',r)}^{a} = 0,$$

where $a = 1, ..., m_{(l+q+w)\text{mod}[2]}$, $z' = (s+q)\text{mod}[2]$, $z'' = (t+w)\text{mod}[2]$,

Condition (1.8) expresses as

$$M_{(i,l)(1,0)}^{1} = \begin{cases} \xi_i^0 & \text{si } l = 0 \text{ et } i \neq 1, \\ 0 & \text{si } l = 1, \\ 1 & \text{si } l = 0 \text{ et } i = 1 \end{cases}$$

and

$$M_{(i,l)(1,0)}^{k} = 0 \text{ if } k \neq 1,$$

where $l = \{0,1\}$, $i = 1, ..., n_l$, $k = 1, ..., m_l$.
Let $f : V \longrightarrow V$ be a morphism which ensure the transportation of the structure. We set, with a basis $\{v_p^q\}_{q=0,1;p=1,...,m_q}$ of V, $f(v_p^q) = \sum_{k=1}^{m_q} T_{(p,q)}^k v_k^q$. Two supermodules, given by their structure constants, are isomorphic if there exist matrices $(T_{(i,0)}^k, T_{(i,1)}^k)_{i,k}$ defining a supermodule morphism with

respect to the basis, that is satisfying $\det(T^k_{(i,0)}) \cdot \det(T^k_{(i,1)}) \neq 0$ and Condition (1.3).

Let $V_1 = (V, \rhd_1)$, $V_2 = (V, \rhd_2)$ be two supermodules. We set, with respect to a basis $\{e^s_i\}_{s=0,1;p=1,...,n_s}$ of A,

$$e^s_i \rhd_1 v^q_p = \sum_{h=1}^{m_r} M^h_{(i,s)(p,q)} v^r_h \quad \text{with } r = (s+q)\text{mod}[2].$$

$$e^s_i \rhd_2 v^q_p = \sum_{h=1}^{m_r} N^h_{(i,s)(p,q)} v^r_h \quad \text{with } r = (s+q)\text{mod}[2].$$

Condition (1.3) expresses as

$$\sum_{k=1}^{m_q} T^k_{(p,q)} M^z_{(i,s)(k,q)} - \sum_{h=1}^{m_r} N^h_{(i,s)(p,q)} T^z_{(h,r)} = 0, \quad \text{where} \quad z = 1, ..., m_r.$$

Two supercomodules, given by their structure constants, are isomorphic if there exist matrices $(T^k_{(i,0)}, T^k_{(i,1)})_{i,k}$ defining a supercomodule morphism with respect to the basis, that is satisfying $\det(T^k_{(i,0)}) \cdot \det(T^k_{(i,1)}) \neq 0$ and Condition (1.6).

Let $V_1 = (V, \rho_1)$, $V_2 = (V, \rho_2)$ be two supercomodules. We set, with respect to a basis $\{e^s_j\}_{s=0,1;j=1,...,n_s}$ of C,

$$\rho_1(v^q_p) = \sum_{s=0}^{1} \sum_{j=1}^{n_s} \sum_{h=1}^{m_r} R^{(j,s)(h)}_{(p,q)} e^s_j \otimes v^r_h \quad \text{with } r = (s+q)\text{mod}[2].$$

$$\rho_2(v^q_p) = \sum_{s=0}^{1} \sum_{j=1}^{n_s} \sum_{h=1}^{m_r} L^{(j,s)(h)}_{(p,q)} e^s_j \otimes v^r_h \quad \text{with } r = (s+q)\text{mod}[2].$$

Condition (1.6) expresses as

$$\sum_{k=1}^{m_q} T^k_{(p,q)} R^{(j,s)(z)}_{(k,q)} - \sum_{h=1}^{m_r} L^{(j,s)(h)}_{(p,q)} T^z_{(h,r)} = 0, \quad \text{with} \quad z = 1, ..., m_r.$$

Notice that the sets of m-dimensional supermodules, supercomodules and supermodule superalgebras provide algebraic varieties given by polynomial equations systems.

They are embedded in the affine space $\mathbb{K}^{n_0 m_0^2 + n_0 m_1^2 + 2 n_0 m_0 m_1}$.

2.1. **Low dimensional supermodules, supercomodules and supermodules superalgebras.** We provide in the following the supermodule, supercomodule and supermodule superalgebra structures over the 2-dimensional Hopf superalgebra $\mathcal{H}$ and the 4-dimensional Hopf superalgebra $\mathcal{H}_0$. The same is done for other Hopf superalgebras $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ and presented in the Appendix (see page 23). Note that all the obtained structures are non-isomorphic.

The superalgebra structure of $\mathcal{H}$ is defined as a quotient $\mathbb{K}[x]/(x^2)$, with $\deg(x) = 1$. Consider a basis $\{1, x\}$ and relations $x^2 = 0$, the Hopf superalgebra $\mathcal{H}$ is defined by the following comultiplication, counit and antipode

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \epsilon(x) = 0, \quad S(x) = -x.$$

2.1.1. **Low dimensional supermodules over $\mathcal{H}$.** Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$. In the sequel, α_1 and α_2 denote parameters in $\mathbb{K}$.

Proposition 2.1. *The supermodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 1,\ m_1 = 0.$ $1 \triangleright v_1^0 = v_1^0$.
- *Case* $m_0 = 0,\ m_1 = 1.$ $1 \triangleright v_1^1 = v_1^1$.

Proposition 2.2. *The supermodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 2,\ m_1 = 0.$ $1 \triangleright v_1^0 = v_1^0, 1 \triangleright v_2^0 = v_2^0$.
- *Case* $m_0 = 0,\ m_1 = 2.$ $1 \triangleright v_1^1 = v_1^1, 1 \triangleright v_2^1 = v_2^1$.
- *Case* $m_0 = 1,\ m_1 = 1.$

(1) $x \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_1^0 = 0,$ (2) $x \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_1^1 = 0.$

Proposition 2.3. *The supermodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 3,\ m_1 = 0.$ $1 \triangleright v_1^0 = v_1^0, 1 \triangleright v_2^0 = v_2^0, 1 \triangleright v_3^0 = v_3^0$.
- *Case* $m_0 = 0,\ m_1 = 3.$ $1 \triangleright v_1^1 = v_1^1, 1 \triangleright v_2^1 = v_2^1, 1 \triangleright v_3^1 = v_3^1$.
- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $x \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_2^1 = \alpha_2 v_1^0, x \triangleright v_1^0 = 0,$ (2) $x \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1, x \triangleright v_1^1 = x \triangleright v_2^1 = 0.$
- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $x \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_2^0 = \alpha_2 v_1^1, x \triangleright v_1^1 = 0,$ (2) $x \triangleright v_1^1 = \alpha_1 v_1^0 + \alpha_2 v_2^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0.$

2.1.2. **Low dimensional supercomodules over $\mathcal{H}$.** Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$. In the sequel, γ_1 and γ_2 denote parameters in $\mathbb{K}$.

Proposition 2.4. *The supercomodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 1,\ m_1 = 0.$ $\rho(v_1^0) = 1 \otimes v_1^0$.
- *Case* $m_0 = 0,\ m_1 = 1.$ $\rho(v_1^1) = 1 \otimes v_1^1$.

Proposition 2.5. *The supercomodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 2,\ m_1 = 0.$ $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0$.
- *Case* $m_0 = 0,\ m_1 = 2.$ $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$.
- *Case* $m_0 = 1,\ m_1 = 1.$

(1) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1 + \gamma_1 x \otimes v_1^0,$ (2) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1.$

Proposition 2.6. *The supercomodule structures over $\mathcal{H}$ are defined in the different cases as follows.*
- *Case* $m_0 = 3,\ m_1 = 0.$ $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_3^0) = 1 \otimes v_3^0$.
- *Case* $m_0 = 0,\ m_1 = 3.$ $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1, \rho(v_3^1) = 1 \otimes v_3^1$.
- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1 + \gamma_1 x \otimes v_1^0, \rho(v_2^1) = 1 \otimes v_2^1 + \gamma_2 x \otimes v_1^0$.
(2) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_1^1 + \gamma_2 x \otimes v_2^1, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$.

• *Case* $m_0 = 2,\ m_1 = 1$.

(1) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0 + \gamma_2 x \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$.
(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1 + \gamma_1 x \otimes v_1^0 + \gamma_2 x \otimes v_2^0$.

2.1.3. **Low dimensional supermodule superalgebras over $\mathcal{H}$.** Let A be a superalgebra of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of A, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$.

Proposition 2.7. *The supermodule superalgebra structures over $\mathcal{H}$ are defined in the different cases as follows.*
• *Case* $m_0 = 1,\ m_1 = 0$. $v_1^0 v_1^0 = v_1^0,\ 1 \triangleright v_1^0 = v_1^0, x \triangleright v_1^0 = 0$.

Proposition 2.8. *The supermodule superalgebra structures over $\mathcal{H}$ are defined in the different cases as follows.*
• *Case* $m_0 = 2,\ m_1 = 0$. (1) $v_2^0 v_2^0 = v_2^0,\ 1 \triangleright v_1^0 = v_1^0, 1 \triangleright v_2^0 = v_2^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0$,
(2) $v_2^0 v_2^0 = 0,\ 1 \triangleright v_1^0 = v_1^0, 1 \triangleright v_2^0 = v_2^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0$.
• *Case* $m_0 = 1,\ m_1 = 1$. $v_1^1 v_1^1 = 0,\ x \triangleright v_1^1 = \lambda_1 v_1^0, x \triangleright v_1^0 = 0$.

Proposition 2.9. *The supermodule superalgebra structures over $\mathcal{H}$ are defined in the different cases as follows.*
• *Case* $m_0 = 3,\ m_1 = 0$. $1 \triangleright v_1^0 = v_1^0, 1 \triangleright v_2^0 = v_2^0, 1 \triangleright v_3^0 = v_3^0,\ x \triangleright v_1^0 = x \triangleright v_2^0 = x \triangleright v_3^0 = 0$.
• *Case* $m_0 = 1,\ m_1 = 2$. $v_1^1 v_1^1 = v_1^1 v_2^1 = v_2^1 v_1^1 = v_2^1 v_2^1 = 0,\ x \triangleright v_1^0 = x \triangleright v_1^1 = x \triangleright v_2^1 = 0$.
• *Case* $m_0 = 2,\ m_1 = 1$. (1) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1 v_1^1 = 0,\ x \triangleright v_1^0 = x \triangleright v_1^1 = 0, x \triangleright v_2^0 = \lambda_1 v_1^1$,
(2) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1 v_1^1 = 0,\ x \triangleright v_1^0 = x \triangleright v_2^0 = 0, x \triangleright v_1^1 = \lambda_1 v_2^0$,
(3) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = 0, v_1^1 v_1^1 = v_2^0,\ x \triangleright v_1^0 = x \triangleright v_2^0 = 0, x \triangleright v_1^1 = \lambda_1 v_2^0$,
(4) $v_2^0 v_2^0 = v_1^1 v_1^1 = v_2^0, v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1,\ x \triangleright v_1^0 = x \triangleright v_2^0 = 0, x \triangleright v_1^1 = \lambda_1 v_2^0$,
(5) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1, v_1^1 v_1^1 = v_1^1 v_2^0 = 0,\ x \triangleright v_1^0 = x \triangleright v_1^1 = 0, x \triangleright v_2^0 = \lambda_1 v_1^1$,
(6) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1, v_1^1 v_1^1 = 0,\ x \triangleright v_1^0 = x \triangleright v_2^0 = 0, x \triangleright v_1^1 = \lambda_1 v_2^0$.

2.1.4. **Low dimensional supermodules over $\mathcal{H}_0$.** The superalgebra structure of $\mathcal{H}_0$ is defined as a quotient $\mathbb{K}[x,y]/(x^2+y^2-1, xy)$, with $\deg(x) = 0$, $\deg(y) = 1$. Consider a basis $\{1, x, x^2, y\}$ and relations $xy = yx = 0,\ y^2 = 1 - x^2,\ x^3 = x$, the Hopf superalgebra $\mathcal{H}_0$ is defined by the following comultiplication, counit and antipode

$$\Delta(x) = x \otimes x - \alpha y \otimes y, \quad \epsilon(x) = 1, \quad S(x) = x,$$
$$\Delta(y) = x \otimes y + y \otimes x, \quad \epsilon(y) = 0, \quad S(y) = \alpha y,$$

where α is a primitive 4^{th} root of unity.

Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$. In the sequel, α_1 and α_2 denote parameters in $\mathbb{K}$.

Proposition 2.10. *The supermodule structures over $\mathcal{H}_0$ are defined in the different cases as follows.*

• *Case* $m_0 = 1,\ m_1 = 0.$
(1) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0$, (2) $x \rhd v_1^0 = -v_1^0, x^2 \rhd v_1^0 = v_1^0$.
• *Case* $m_0 = 0,\ m_1 = 1.$
(1) $x \rhd v_1^1 = x^2 \rhd v_1^1 = v_1^1$, (2) $x \rhd v_1^1 = -v_1^1, x^2 \rhd v_1^1 = v_1^1$.

Proposition 2.11. *The supermodule structures over* $\mathcal{H}_0$ *are defined in the different cases as follows.*
• *Case* $m_0 = 2,\ m_1 = 0.$

(1) $x \rhd v_1^0 = v_1^0 + \alpha_1 v_2^0, x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = -v_2^0, x^2 \rhd v_2^0 = v_2^0$,
(2) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = \alpha_1 v_1^0 - v_2^0, x^2 \rhd v_2^0 = v_2^0$,
(3) $x \rhd v_1^0 = -v_1^0, x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = \alpha_1 v_1^0 + v_2^0, x^2 \rhd v_2^0 = v_2^0$,
(4) $x \rhd v_1^0 = -v_1^0, x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = -v_2^0, x^2 \rhd v_2^0 = v_2^0$,
(5) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = x^2 \rhd v_2^0 = v_2^0$.

• *Case* $m_0 = 0,\ m_1 = 2.$

(1) $x \rhd v_1^1 = x^2 \rhd v_1^1 = v_1^1, x \rhd v_2^1 = \alpha_1 v_1^1 - v_2^1, x^2 \rhd v_2^1 = v_2^1$,
(2) $x \rhd v_1^1 = -v_1^1, x^2 \rhd v_1^1 = v_1^1, x \rhd v_2^1 = -v_2^1, x^2 \rhd v_2^1 = v_2^1$,
(3) $x \rhd v_1^1 = x^2 \rhd v_1^1 = v_1^1, x \rhd v_2^1 = x^2 \rhd v_2^1 = v_2^1$.

• *Case* $m_0 = 1,\ m_1 = 1.$

(1) $y \rhd v_1^0 = \alpha_1 v_1^1, y \rhd v_1^1 = \frac{1}{\alpha_1} v_1^0, x \rhd v_1^0 = x \rhd v_1^1 = x^2 \rhd v_1^0 = x^2 \rhd v_1^1 = 0, (\alpha_1 \neq 0)$,
(2) $x \rhd v_1^0 = -v_1^0, x \rhd v_1^1 = -v_1^1, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_1^1 = v_1^1, y \rhd v_1^0 = y \rhd v_1^1 = 0$,
(3) $x \rhd v_1^0 = -v_1^0, x \rhd v_1^1 = x^2 \rhd v_1^1 = v_1^1, x^2 \rhd v_1^0 = v_1^0, y \rhd v_1^0 = y \rhd v_1^1 = 0$,
(4) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_1^1 = -v_1^1, x^2 \rhd v_1^1 = v_1^1, y \rhd v_1^0 = y \rhd v_1^1 = 0$,
(5) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_1^1 = x^2 \rhd v_1^1 = v_1^1, y \rhd v_1^0 = y \rhd v_1^1 = 0$.

Proposition 2.12. *The supermodule structures over* $\mathcal{H}_0$ *are defined in the different cases as follows.*
• *Case* $m_0 = 3,\ m_1 = 0.$

(1) $x \rhd v_1^0 = -v_1^0, x \rhd v_2^0 = -v_2^0, x \rhd v_3^0 = \alpha_1 v_1^0 + \alpha_2 v_2^0 + v_3^0, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_2^0 = v_2^0$,
$x^2 \rhd v_3^0 = v_3^0$,
(2) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = x^2 \rhd v_2^0 = v_2^0, x^2 \rhd v_3^0 = v_3^0, x \rhd v_3^0 = \alpha_1 v_1^0 + \alpha_2 v_2^0 - v_3^0$,
(3) $x \rhd v_1^0 = v_1^0 + \alpha_1 v_2^0, x \rhd v_2^0 = -v_2^0, x \rhd v_3^0 = \alpha_2 v_2^0 + v_3^0, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_2^0 = v_2^0, x^2 \rhd v_3^0 = v_3^0$
(4) $x \rhd v_1^0 = -v_1^0 + \alpha_1 v_2^0, x \rhd v_2^0 = x^2 \rhd v_2^0 = v_2^0, x \rhd v_3^0 = \alpha_2 v_2^0 - v_3^0, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_3^0 = v_3^0$,
(5) $x \rhd v_1^0 = -v_1^0, x \rhd v_2^0 = \alpha_1 v_1^0 + v_2^0, x \rhd v_3^0 = \alpha_2 v_1^0 + v_3^0, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_2^0 = v_2^0, x^2 \rhd v_3^0 = v_3^0$
(6) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = \alpha_1 v_1^0 - v_2^0, x \rhd v_3^0 = \alpha_2 v_1^0 - v_3^0, x^2 \rhd v_2^0 = v_2^0, x^2 \rhd v_3^0 = v_3^0$,
(7) $x \rhd v_1^0 = -v_1^0, x \rhd v_2^0 = -v_2^0, x \rhd v_3^0 = -v_3^0, x^2 \rhd v_1^0 = v_1^0, x^2 \rhd v_2^0 = v_2^0, x^2 \rhd v_3^0 = v_3^0$,
(8) $x \rhd v_1^0 = x^2 \rhd v_1^0 = v_1^0, x \rhd v_2^0 = x^2 \rhd v_2^0 = v_2^0, x \rhd v_3^0 = x^2 \rhd v_3^0 = v_3^0$.

• *Case* $m_0 = 0,\ m_1 = 3.$

(1) $x \rhd v_1^1 = v_1^1 + \alpha_1 v_2^1, x \rhd v_2^1 = -v_2^1, x \rhd v_3^1 = \alpha_2 v_2^1 + v_3^1, x^2 \rhd v_1^1 = v_1^1, x^2 \rhd v_2^1 = v_2^1, x^2 \rhd v_3^1 = v_3^1$

(2) $x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1 - v_2^1, x \triangleright v_3^1 = \alpha_2 v_1^1 - v_3^1, x^2 \triangleright v_2^1 = v_2^1, x^2 \triangleright v_3^1 = v_3^1$

(3) $x \triangleright v_1^1 = -v_1^1, x \triangleright v_2^1 = -v_2^1, x \triangleright v_3^1 = -v_3^1, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1, x^2 \triangleright v_3^1 = v_3^1,$

(4) $x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = x^2 \triangleright v_2^1 = v_2^1, x \triangleright v_3^1 = x^2 \triangleright v_3^1 = v_3^1.$

- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $y \triangleright v_1^0 = \alpha_1\alpha_2 v_1^1 + \alpha_1 v_2^1, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_2 v_1^1, x^2 \triangleright v_2^1 = -\alpha_2 v_1^1,$
$y \triangleright v_2^1 = \frac{1}{\alpha_1} v_1^0, x \triangleright v_1^0 = x^2 \triangleright v_1^0 = y \triangleright v_1^1 = 0, (\alpha_1 \neq 0),$

(2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_1^0 = -\alpha_1\alpha_2 v_1^1 + \alpha_2 v_2^1,$
$y \triangleright v_2^1 = \frac{1}{\alpha_2} v_1^0, x \triangleright v_1^0 = x^2 \triangleright v_1^0 = y \triangleright v_1^1 = 0, (\alpha_2 \neq 0),$

(3) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_1^1 = -v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1 + v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0,$

(4) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = -v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1 + v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0,$

(5) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_1^1 = -v_1^1, x \triangleright v_2^1 = -v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1, y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0,$

(6) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = -v_1^1, x \triangleright v_2^1 = -v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0,$

(7) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0,$

(8) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0.$

- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = -\alpha_1 v_1^0, y \triangleright v_2^0 = \frac{1}{\alpha_2} v_1^1,$
$y \triangleright v_1^1 = \alpha_1\alpha_2 v_1^0 + \alpha_1 v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = y \triangleright v_1^0 = 0, (\alpha_2 \neq 0),$

(2) $x \triangleright v_2^0 = -v_2^0, x^2 \triangleright v_2^0 = v_2^0, y \triangleright v_1^0 = \frac{1}{\alpha_1} v_1^1, y \triangleright v_1^1 = \alpha_1 v_1^0,$
$x \triangleright v_1^0 = x \triangleright v_1^1 = x^2 \triangleright v_1^0 = x^2 \triangleright v_1^1 = y \triangleright v_2^0 = 0, (\alpha_1 \neq 0),$

(3) $x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, y \triangleright v_1^0 = \frac{1}{\alpha_1} v_1^1, y \triangleright v_1^1 = \alpha_1 v_1^0,$
$x \triangleright v_1^0 = x \triangleright v_1^1 = x^2 \triangleright v_1^0 = x^2 \triangleright v_1^1 = y \triangleright v_2^0 = 0, (\alpha_1 \neq 0),$

(4) $x \triangleright v_1^0 = x^2 \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = x^2 \triangleright v_2^0 = \alpha_1 v_1^0, y \triangleright v_2^0 = \frac{1}{\alpha_2} v_1^1,$
$y \triangleright v_1^1 = \alpha_1\alpha_2 v_1^0 + \alpha_2 v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = y \triangleright v_1^0 = 0, (\alpha_2 \neq 0),$

(5) $x \triangleright v_1^0 = \alpha_1 v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{1-\alpha_1^2}{\alpha_2} v_1^0 - \alpha_1 v_2^0, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^0 = v_1^0,$
$x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0, (\alpha_2 \neq 0),$

(6) $x \triangleright v_1^0 = \alpha_1 v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{1-\alpha_1^2}{\alpha_2} v_1^0 - \alpha_1 v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_1^0 = v_1^0,$
$x^2 \triangleright v_2^0 = v_2^0, y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0, (\alpha_2 \neq 0),$

(7) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(8) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(9) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0 - v_2^0, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(10) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0 - v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(11) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_2^0 = -v_2^0, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0,$
$x^2 \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(12) $x \triangleright v_1^0 = -v_1^0, x \triangleright v_2^0 = -v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_1^0 = v_1^0, x^2 \triangleright v_2^0 = v_2^0,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(13) $x \triangleright v_1^0 = x^2 \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0,$
(14) $x \triangleright v_1^0 = x^2 \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = 0.$

2.1.5. **Low dimensional supercomodules over $\mathcal{H}_0$.** Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$. In the sequel, γ_1 and γ_2 denote parameters in $\mathbb{K}$.

Proposition 2.13. *The supercomodule structures over $\mathcal{H}_0$ are defined in the different cases as follows.*
• *Case* $m_0 = 1,\ m_1 = 0.$
(1) $\rho(v_1^0) = 1 \otimes v_1^0,$ (2) $\rho(v_1^0) = (-1 + 2x^2) \otimes v_1^0.$
• *Case* $m_0 = 0,\ m_1 = 1.$ $\rho(v_1^1) = 1 \otimes v_1^1,\ \rho(v_1^1) = (-1 + 2x^2) \otimes v_1^1.$

Proposition 2.14. *The supercomodule structures over $\mathcal{H}_0$ are defined in the different cases as follows.*
• *Case* $m_0 = 2,\ m_1 = 0.$
(1) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1(1 - x^2) \otimes v_2^0, \rho(v_2^0) = (-1 + 2x^2) \otimes v_2^0,$
(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \gamma_1(1 - x^2) \otimes v_1^0 + (-1 + 2x^2) \otimes v_2^0,$
(3) $\rho(v_1^0) = (-1 + 2x^2) \otimes v_1^0, \rho(v_2^0) = \gamma_1(1 - x^2) \otimes v_1^0 + 1 \otimes v_2^0,$
(4) $\rho(v_1^0) = (-1 + 2x^2) \otimes v_1^0, \rho(v_2^0) = (-1 + 2x^2) \otimes v_2^0,$
(5) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0.$

• *Case* $m_0 = 0,\ m_1 = 2.$
(1) $\rho(v_1^1) = 1 \otimes v_1^1 + \gamma_1(1 - x^2) \otimes v_2^1, \rho(v_2^1) = (-1 + 2x^2) \otimes v_2^1,$
(2) $\rho(v_1^1) = (-1 + 2x^2) \otimes v_1^1, \rho(v_2^1) = (-1 + 2x^2) \otimes v_2^1,$
(3) $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1.$

• *Case* $m_0 = 1,\ m_1 = 1.$
(1) $\rho(v_1^0) = x \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = -\frac{\alpha}{\gamma_1} y \otimes v_1^0 + x \otimes v_1^1, (\gamma_1 \neq 0),$
(2) $\rho(v_1^0) = (-1 + 2x^2) \otimes v_1^0, \rho(v_1^1) = (-1 + 2x^2) \otimes v_1^1,$

(3) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1,$
(4) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (-1+2x^2) \otimes v_1^1,$
(5) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1.$

Proposition 2.15. *The supercomodule structures over $\mathcal{H}_0$ are defined in the different cases as follows.*

- *Case* $m_0 = 3,\ m_1 = 0.$
  (1) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (-1+2x^2) \otimes v_2^0, \rho(v_3^0) = \big(\gamma_1(1-x^2)\big) \otimes v_2^0 + 1 \otimes v_3^0,$
  (2) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_2^0) = \big(\gamma_1(1-x^2)\big) \otimes v_1^0 + 1 \otimes v_2^0 + \big(\gamma_2(1-x^2)\big) \otimes v_3^0,$
  $\rho(v_3^0) = (-1+2x^2) \otimes v_3^0,$
  (3) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \big(\gamma_1(1-x^2)\big) \otimes v_1^0 + (-1+2x^2) \otimes v_2^0,$
  $\rho(v_3^0) = \big(\gamma_2(1-x^2)\big) \otimes v_1^0 + (-1+2x^2) \otimes v_3^0,$
  (4) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0 + \big(\gamma_1(1-x^2)\big) \otimes v_2^0,$
  $\rho(v_2^0) = 1 \otimes v_2^0, \rho(v_3^0) = \big(\gamma_2(1-x^2)\big) \otimes v_2^0 + (-1+2x^2) \otimes v_3^0,$
  (5) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_2^0) = \big(\gamma_1(1-x^2)\big) \otimes v_1^0 + 1 \otimes v_2^0,$
  $\rho(v_3^0) = \big(\gamma_2(1-x^2)\big) \otimes v_1^0 + 1 \otimes v_3^0,$
  (6) $\rho(v_1^0) = 1 \otimes v_1^0 + \big(\gamma_1(1-x^2)\big) \otimes v_2^0, \rho(v_2^0) = (-1+2x^2) \otimes v_2^0,$
  $\rho(v_3^0) = \big(\gamma_2(1-x^2)\big) \otimes v_2^0 + 1 \otimes v_3^0,$
  (7) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_2^0) = (-1+2x^2) \otimes v_2^0, \rho(v_3^0) = (-1+2x^2) \otimes v_3^0,$
  (8) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_3^0) = 1 \otimes v_3^0.$
- *Case* $m_0 = 0,\ m_1 = 3.$
  (1) $\rho(v_1^1) = (-1+2x^2) \otimes v_1^1, \rho(v_2^1) = \big(\gamma_1(1-x^2)\big) \otimes v_1^1 + 1 \otimes v_2^1,$
  $\rho(v_3^1) = \big(\gamma_2(1-x^2)\big) \otimes v_1^1 + 1 \otimes v_3^1,$
  (2) $\rho(v_1^1) = (-1+2x^2) \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1,$
  $\rho(v_3^0) = \big(\gamma_1(1-x^2)\big) \otimes v_1^1 + \big(\gamma_2(1-x^2)\big) \otimes v_2^1 + 1 \otimes v_3^1,$
  (3) $\rho(v_1^1) = (-1+2x^2) \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1, \rho(v_3^1) = (-1+2x^2) \otimes v_3^1,$
  (4) $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1, \rho(v_3^1) = 1 \otimes v_3^1.$
- *Case* $m_0 = 1,\ m_1 = 2.$
  (1) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1 + \big(\gamma_1(1-x^2)\big) \otimes v_1^1,$
  (2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = \big(\gamma_1(1-x^2)\big) \otimes v_1^1 + (-1+2x^2) \otimes v_2^1,$
  (3) $\rho(v_1^0) = x \otimes v_1^0 - \frac{\alpha}{\gamma_1} y \otimes v_1^1, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + x \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1, (\gamma_1 \neq 0),$
  (4) $\rho(v_1^0) = x \otimes v_1^0 - \frac{\alpha}{\gamma_1} y \otimes v_1^1, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + x \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1, (\gamma_1 \neq 0),$
  (5) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_1^1) = (-1+2x^2) \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1,$
  (6) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (-1+2x^2) \otimes v_1^1, \rho(v_2^1) = (-1+2x^2) \otimes v_2^1,$
  (7) $\rho(v_1^0) = (-1+2x^2) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1,$
  (8) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1.$
- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $\rho(v_1^0) = x\otimes v_1^0 + \gamma_1(1+x-2x^2)\otimes v_2^0 - \frac{\alpha}{\gamma_2} y\otimes v_1^1, \rho(v_2^0) = (-1+2x^2)\otimes v_2^0,$
$\rho(v_1^1) = \gamma_2 y\otimes v_1^0 + \gamma_1\gamma_2 y\otimes v_2^0 + x\otimes v_1^1, (\gamma_2 \neq 0),$
(2) $\rho(v_1^0) = x\otimes v_1^0 + \gamma_1(1+x)\otimes v_2^0 - \gamma_1\gamma_2 y\otimes v_1^1, \rho(v_2^0) = x\otimes v_2^0 + \gamma_2 y\otimes v_1^1$
$\rho(v_1^1) = -\frac{\alpha}{\gamma_2} y\otimes v_2^0 + x\otimes v_1^1, (\gamma_2 \neq 0),$
(3) $\rho(v_1^0) = (-1+2x^2)\otimes v_1^0, \rho(v_2^0) = \gamma_1(1+x-2x^2)\otimes v_1^0 + x\otimes v_2^0 + \gamma_2 y\otimes v_1^1,$
$\rho(v_1^1) = -\frac{\alpha\gamma_1}{\gamma_2} y\otimes v_1^0 - \frac{\alpha}{\gamma_2} y\otimes v_2^0 + x\otimes v_1^1, (\gamma_2 \neq 0),$
(4) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = \gamma_1(1-x)\otimes v_1^0 + x\otimes v_2^0 + \gamma_2 y\otimes v_1^1,$
$\rho(v_1^1) = \frac{\alpha\gamma_1}{\gamma_2} y\otimes v_1^0 - \frac{\alpha}{\gamma_2} y\otimes v_2^0 + x\otimes v_1^1, (\gamma_2 \neq 0),$
(5) $\rho(v_1^0) = (-1+2x^2)\otimes v_1^0, \rho(v_2^0) = 1\otimes v_2^0, \rho(v_1^1) = (-1+2x^2)\otimes v_1^1,$
(6) $\rho(v_1^0) = 1\otimes v_1^0 + \gamma_1(1-x^2)\otimes v_2^0, \rho(v_2^0) = (-1+2x^2)\otimes v_2^0, \rho(v_1^1) = 1\otimes v_1^1,$
(7) $\rho(v_1^0) = (-1+2x^2)\otimes v_1^0, \rho(v_2^0) = (-1+2x^2)\otimes v_2^0, \rho(v_1^1) = (-1+2x^2)\otimes v_1^1,$
(8) $\rho(v_1^0) = (-1+2x^2)\otimes v_1^0, \rho(v_2^0) = (-1+2x^2)\otimes v_2^0, \rho(v_1^1) = 1\otimes v_1^1,$
(9) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = 1\otimes v_2^0, \rho(v_1^1) = (-1+2x^2)\otimes v_1^1,$
(10) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = 1\otimes v_2^0, \rho(v_1^1) = 1\otimes v_1^1.$

2.1.6. ***Low dimensional supermodule superalgebras over*** $\mathcal{H}_0$. Let A be a superalgebra of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$be a basis of A, where $q \in \{0,1\}, p \in \{1,\cdots,m_q\}$. In the sequel, λ_1 denote a parameter in $\mathbb{K}$.

Proposition 2.16. *The supermodule superalgebra structures over $\mathcal{H}_0$ are defined in the different cases as follows.*
• *Case $m_0 = 1,\ m_1 = 0$.* $v_1^0 v_1^0 = v_1^0,\ x \triangleright v_1^0 = x^2 \triangleright v_1^0 = v_1^0.$

Proposition 2.17. *The supermodule superalgebra structures over $\mathcal{H}_0$ are defined in the different cases as follows.*
• *Case $m_0 = 2,\ m_1 = 0$.*
(1) $v_2^0 v_2^0 = 0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0,$ (2) $v_2^0 v_2^0 = 0,\ x \triangleright v_2^0 = -v_2^0, x^2 \triangleright v_2^0 = v_2^0,$
(3) $v_2^0 v_2^0 = v_2^0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0,$ (4) $v_2^0 v_2^0 = v_2^0,\ x \triangleright v_2^0 = v_1^0 - v_2^0, x^2 \triangleright v_2^0 = v_2^0,$
• *Case $m_0 = 1,\ m_1 = 1$.* $v_1^1 v_1^1 = 0,$
(1) $x \triangleright v_1^1 = -v_1^1, x^2 \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = 0,$ (2) $x \triangleright v_1^1 = x^2 \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = 0.$

Proposition 2.18. *The supermodule superalgebra structures over $\mathcal{H}_0$ are defined in the different cases as follows.*
• *Case $m_0 = 3,\ m_1 = 0$.*
(1) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0 v_3^0 = v_3^0,\ x \triangleright v_2^0 = v_1^0 - v_2^0 + v_3^0, x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = x^2 \triangleright v_3^0 = v_3^0,$
(2) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0 v_3^0 = v_3^0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = x^2 \triangleright v_3^0 = v_3^0,$
(3) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0, v_3^0 v_3^0 = 0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = -v_3^0, x^2 \triangleright v_3^0 = v_3^0,$
(4) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0, v_3^0 v_3^0 = 0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = x^2 \triangleright v_3^0 = v_3^0,$
(5) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0 v_3^0 = 0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = -v_3^0, x^2 \triangleright v_3^0 = v_3^0,$
(6) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0 v_3^0 = 0,\ x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 =$

$x^2 \triangleright v_3^0 = v_3^0$,
(7) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \lambda_1 v_2^0 - v_3^0, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_3^0 = v_3^0$,
(8) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0$, $x \triangleright v_2^0 = av_2^0, x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = av_3^0, x^2 \triangleright v_3^0 = v_3^0$,
with $a \in \{-1, 1\}$,
(9) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0, v_3^0v_2^0 = v_3^0v_3^0 = 0$, $x \triangleright v_2^0 = v_2^0 + \lambda_1 v_3^0, x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = -v_3^0$,
$x^2 \triangleright v_3^0 = v_3^0$,
(10) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0, v_3^0v_2^0 = v_3^0v_3^0 = 0$, $x \triangleright v_2^0 = x^2 \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = x^2 \triangleright v_3^0 = v_3^0$.

- *Case* $m_0 = 1,\ m_1 = 2$. $v_1^1v_1^1 = v_1^1v_2^1 = v_2^1v_1^1 = v_2^1v_2^1 = 0$,

(1) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \lambda_1 v_1^1 - v_2^1, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = y \triangleright v_2^1 = 0$,
(2) $x \triangleright v_1^1 = av_1^1, x \triangleright v_2^1 = av_2^1, x^2 \triangleright v_1^1 = v_1^1, x^2 \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = y \triangleright v_2^1 = 0$,
with $a \in \{-1, 1\}$,

- *Case* $m_0 = 2,\ m_1 = 1$. (1) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0$, $x \triangleright v_2^0 = x^2 \triangleright v_2^0 = x \triangleright v_1^1 = x^2 \triangleright v_1^1 = 0, y \triangleright v_2^0 = \frac{1}{\lambda} v_1^1, y \triangleright v_1^1 = \lambda v_2^0$, *with* $\lambda \in \mathbb{K}^*$

(2) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0$, $x \triangleright v_2^0 = -v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$,
(3) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$,
(4) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = 0, v_1^1v_1^1 = v_2^0$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$,
(5) $v_2^0v_2^0 = v_1^1v_1^1 = v_2^0, v_0^2v_1^1 = v_1^1v_2^0 = v_1^1$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$,
(6) $v_2^0v_2^0 = v_2^0, v_2^0v_1^1 = v_1^1, v_1^1v_1^1 = v_1^1v_2^0 = 0$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$,
(7) $v_2^0v_2^0 = v_2^0, v_2^0v_1^1 = v_1^1v_2^0 = v_1^1, v_1^1v_1^1 = 0$, $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = av_1^1, x^2 \triangleright v_2^0 = v_2^0, x^2 \triangleright v_1^1 = v_1^1$,
$y \triangleright v_2^0 = y \triangleright v_1^1 = 0$, *with* $a \in \{-1, 1\}$.

3. Twistings on superbialgebras

In this section, we provide a construction of a new superalgebra by twisting, based on Drinfeld's twists and following the study in algebras case by Giaquinto and Zhang in [12]. The approach is illustrated by an example at the end. Let $(B, \mu_B, \eta_B, \Delta_B, \varepsilon_B)$ be a superbialgebra.

Definition 3.1. *An element* $F \in B \otimes B$ *is a right twisting element if the three identities*

$$(\Delta \otimes id)(F)(F \otimes 1) = (id \otimes \Delta)(F)(1 \otimes F), \tag{3.1}$$

$$(\varepsilon \otimes id)(F) = 1 \otimes 1_B \quad \text{and} \quad (id \otimes \varepsilon)(F) = 1_B \otimes 1 \tag{3.2}$$

hold.

Remark 3.2. (1) *A twisting (left twisting) element of B is an element $F \in B \otimes B$ which satisfies*

$$(F \otimes 1)(\Delta \otimes id)(F) = (1 \otimes F)(id \otimes \Delta)(F),$$
$$(\varepsilon \otimes id)(F) = 1 \otimes 1_B \text{ and } (id \otimes \varepsilon)(F) = 1_B \otimes 1.$$

(2) *A right twisting element is also called a Drinfeld twist.*
(3) *In this case, multiplication of tensor products is $\mathbb{Z}_2$-graded and defined as $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd, \quad \forall a, b, c, d \in B$.*

Lemma 3.3. *If F is a Drinfeld twist of a superbialgebra B, then F belongs to the even part of $B \otimes B$.*

Proof. Let F be a twisting element. The duality between Drinfeld twist F and 2-cocycle, and the evenly of 2-cocycle imply that F belongs to the even part of $B \otimes B$. □

We aim to show in the sequel a generalization of a theorem given by Giaquinto and Zhang in [12] into $\mathbb{Z}_2$-graded case. A twisting element provides a new multiplication of any B-supermodule superalgebra. The twisted multiplication is defined to be the composite $\mu_A \circ F_l : A \otimes A \longrightarrow A$.

Theorem 3.4. *Let F be a twisting element and A is a B-supermodule superalgebra. Then $A_F = (A, \mu_F, \eta_A)$ is a superalgebra, where the multiplication $\mu_F = \mu_A \circ F_l$ is defined as*

$$\mu_F(a \otimes b) = (\mu_A \circ F_l)(a \otimes b) = (-1)^{|F_2||a|} \mu_A[(F_1 \rhd a) \otimes (F_2 \rhd b)], \quad \forall a, b \in A.$$

Proof. We show that the multiplication $\mu_A \circ F_l$ is associative, which means

$$(\mu_A \circ F_l) \circ [(\mu_A \circ F_l) \otimes id_A] = (\mu_A \circ F_l) \circ [id_A \otimes (\mu_A \circ F_l)].$$

Let a, b, $c \in A$, by Definitions 1.1 and 1.7, we have

$$\begin{aligned}
(\text{l.h.s}) &= (\mu_A \circ F_l) \circ [(\mu_A \circ F_l) \otimes id_A](a \otimes b \otimes c) \\
&= (-1)^{|F_2||a|} (\mu_A \circ F_l) \circ [(F_1 \rhd a) \cdot (F_2 \rhd b) \otimes c] \\
&= (-1)^{|F_2||b|} F_1 \rhd [(F_1 \rhd a) \cdot (F_2 \rhd b)] \cdot (F_2 \rhd c) \\
&= (-1)^{|F_2||b| + |F_{1(2)}|(|F_1| + |a|)} [F_{1(1)} \rhd (F_1 \rhd a)] \cdot [F_{1(2)} \rhd (F_2 \rhd b)] \cdot (F_2 \rhd c) \\
&= (-1)^{|F_2||b| + |F_{1(2)}|(|F_1| + |a|)} (F_{1(1)} F_1 \rhd a) \otimes (F_{1(2)} F_2 \rhd b) \otimes (F_2 \rhd c) \\
&= [(-1)^{|F_1||F_{1(2)}|} F_{1(1)} F_1 \otimes F_{1(2)} F_2 \otimes F_2]_l (a \otimes b \otimes c) \\
&= [(\Delta \otimes id)(F)(F \otimes 1)]_l (a \otimes b \otimes c).
\end{aligned}$$

$$\begin{aligned}
(\text{r.h.s}) &= (\mu_A \circ F_l) \circ [id_A \otimes (\mu_A \circ F_l)](a \otimes b \otimes c) \\
&= (-1)^{|F_2||b|} (\mu_A \circ F_l) \circ [a \otimes (F_1 \rhd b) \cdot (F_2 \rhd c)] \\
&= (-1)^{|F_2|(|a| + |b|)} (F_1 \rhd a) \cdot F_2 \rhd [(F_1 \rhd b) \cdot (F_2 \rhd c)] \\
&= (-1)^{|F_2|(|a| + |b|) + |F_{2(2)}|(|F_1| + |b|)} (F_1 \rhd a) \cdot [F_{2(1)} \rhd (F_1 \rhd b)] \cdot [F_{2(2)} \rhd (F_2 \rhd c)] \\
&= (-1)^{|F_2|(|a| + |b|) + |F_{2(2)}|(|F_1| + |b|)} (F_1 \rhd a) \cdot (F_{2(1)} F_1 \rhd b) \cdot (F_{2(2)} F_2 \rhd c) \\
&= (-1)^{|a|(|F_1| + |F_2|)} [(-1)^{|F_1||F_{2(2)}|} F_1 \otimes F_{2(1)} F_1 \otimes F_{2(2)} F_2]_l (a \otimes b \otimes c) \\
&= [(-1)^{|F_1||F_{2(2)}|} F_1 \otimes F_{2(1)} F_1 \otimes F_{2(2)} F_2]_l (a \otimes b \otimes c), \quad \text{since} \quad F \in (B \otimes B)_0 \\
&= [(id \otimes \Delta)(F)(1 \otimes F)]_l (a \otimes b \otimes c).
\end{aligned}$$

Since F is a twisting element then $(\mu_A \circ F_l) \circ [(\mu_A \circ F_l) \otimes id_A] = (\mu_A \circ F_l) \circ [id_A \otimes (\mu_A \circ F_l)]$, and thus $\mu_A \circ F_l$ is associative.

For the unit element 1_A, by Definition 3.2, we have

$$(\mu_A \circ F)(1_A \otimes a) = \mu_A \circ [(F_1 \triangleright 1_A) \otimes (F_2 \triangleright a)] = \mu_A \circ [\varepsilon_B(F_1)1_A \otimes (F_2 \triangleright a)]$$
$$= \mu_A \circ [[(\varepsilon_B \otimes id)(F)(1_B \otimes 1)]_l(1_A \otimes a)] = \mu_A \circ [(1_B \otimes 1_B)_l(1_A \otimes a)]$$
$$= \mu_A(1_A \otimes a) = a.$$

Similarly, the unit element 1_A also serves as the right unit with respect to $\mu_A \circ F_l$. Indeed,

$$(\mu_A \circ F)(a \otimes 1_A) = \mu_A \circ ((-1)^{|a||F_2|}(F_1 \triangleright a) \otimes (F_2 \triangleright 1_A))$$
$$= \mu_A \circ [(-1)^{|a||F_2|}(F_1 \triangleright a) \otimes \varepsilon_B(F_2)1_A]$$
$$= \mu_A \circ [(F_1 \triangleright a) \otimes \varepsilon_B(F_2)1_A] = \mu_A \circ [[(id \otimes \varepsilon_B)(F)(1 \otimes 1_B)]_l(a \otimes 1_A)]$$
$$= \mu_A \circ [(1_B \otimes 1_B)_l(a \otimes 1_A)] = \mu_A(a \otimes 1_A) = a.$$

□

Dually we have the following result.

Proposition 3.5. *Let $(B, \mu, 1_B, \Delta, \varepsilon)$ be a superbialgebra over $\mathbb{K}$ and suppose that F is an invertible twisting element based on a superbialgebra B. Then $B_F = (B, \mu, 1, \Delta_F, \varepsilon)$ is a superbialgebra over $\mathbb{K}$, where the comultiplication Δ_F is defined as*

$$\Delta_F = F^{-1}\Delta F.$$

Proof. We show that $(B, \Delta_F, \varepsilon)$ is a supercoalgebra over $\mathbb{K}$. Let $a \in B$, we have

$$(\Delta_F \otimes id_B) \circ \Delta_F(a) =$$
$$(-1)^{\iota}[(F^{-1} \otimes 1)(\Delta \otimes id)(F^{-1})][(\Delta \otimes id_B) \circ \Delta(a)][(\Delta \otimes id)(F)(F \otimes 1)],$$

and

$$(id_C \otimes \Delta_F) \circ \Delta_F(a) =$$
$$(-1)^{\iota}[(1 \otimes F^{-1})(id \otimes \Delta)(F^{-1})][(id_B \otimes \Delta) \circ \Delta(a)][(id \otimes \Delta)(F)(1 \otimes F)],$$

where $\iota = |F_1^{-1}|(|a_1| + |a_2| + |F_1|) + |F_1||a_3|$, the coassociativity of Δ_F is due to F being a Drinfeld twist for B and F^{-1} being a left twisting for B. The counit axiom is satisfied thanks to Condition (3.2), then $(B, \Delta_F, \varepsilon)$ is a supercoalgebra over $\mathbb{K}$. Moreover the comultiplication Δ_F is compatible with μ and the counit ε is a superalgebra morphism. Hence B_F is a superbialgebra. □

Proposition 3.6. *Let B be a superbialgebra over $\mathbb{K}$ and A be a superalgebra. Suppose that F is an invertible twisting element based on a superbialgebra B. If A is a left B-supermodule superalgebra then A_F, defined in Theorem 3.4, is a B_F-supermodule superalgebra (B_F being defined in Proposition 3.5).*

Proof. The multiplications of B and B_F are identical, then A_F is a B_F-supermodule. Condition (1.8) is satisfied since the counits of both B and

B_F are the same. It remains to show Condition (1.7). Let $a, a' \in A$ and $b \in B$,

$$
\begin{aligned}
&(\mu_A \circ F_l) \circ (\Delta_F(b))_l(a \otimes a') \\
&= (-1)^{|F_2^{-1}||b_{(1)}|+|F_1|(|F_2^{-1}|+|b_{(2)}|)}(\mu_A \circ F_l) \circ (F_1^{-1}b_{(1)}F_1 \otimes F_2^{-1}b_{(2)}F_2)_l(a \otimes a') \\
&= (-1)^{|F_1||b_{(2)}|}(\mu_A \circ F_l) \circ [(F_1^{-1} \otimes F_2^{-1})(b_{(1)}F_1 \otimes b_{(2)}F_2)]_l(a \otimes a') \\
&= (-1)^{|F_1||b_{(2)}|}\mu_A \circ [(F_1 \otimes F_1)(F_1^{-1} \otimes F_2^{-1})(b_{(1)}F_1 \otimes b_{(2)}F_2)]_l(a \otimes a') \\
&= (-1)^{|F_1||b_{(2)}|}\mu_A \circ (b_{(1)}F_1 \otimes b_{(2)}F_2)_l(a \otimes a') \\
&= \mu_A \circ [(b_{(1)} \otimes b_{(2)})(F_1 \otimes F_2)]_l(a \otimes a') \\
&= (\mu_A \circ \Delta_F(b)_l) \circ F_l(a \otimes a') \\
&= (b_l \circ \mu_A) \circ F_l(a \otimes a'), \text{ since A is a B-supermodule superalgebra} \\
&= b_l \circ (\mu_A \circ F_l)(a \otimes a').
\end{aligned}
$$

Then A_F is a B_F-supermodule superalgebra. □

Remark 3.7. *If B is a commutative superbialgebra, then the twist superbialgebra B_F is just B itself.*

Now, we consider our Hopf superalgebras up to dimension 4 and compute their twisting elements of the form:

$$F = u_1^0 \otimes u_1^0 + \sum_{l=0}^{1}\sum_{i=1}^{n_l}\sum_{s=0}^{1}\sum_{j=1}^{n_s} \alpha_{(i,l)(j,s)} u_i^l \otimes u_j^s. \tag{3.3}$$

where $\{u_i^0, u_i^1\}_i$ is the basis. By straightforward calculations, we obtain the following results.

Proposition 3.8 (Twisting of 2-dimensional Hopf superalgebra)**.** *The twisting elements of $\mathcal{H}$ are given by $F = 1 \otimes 1 + \beta x \otimes x$, with $\beta \in \mathbb{K}$.*

Proposition 3.9 (Twistings of 4-dimensional Hopf superalgebras)**.**

- *The element $1 \otimes 1$ is the only twisting element of $\mathcal{H}_0$.*
- *The twisting elements of $\mathcal{H}_1$ are given by*
  (1) $F = 1 \otimes 1 + \beta x \otimes x$, (2) $F = 1 \otimes 1 + \beta y \otimes y$, *with* $\beta \in \mathbb{K}$.
- *The twisting elements of $\mathcal{H}_2$ are given by $F = 1 \otimes 1 + \beta x \otimes x$, with $\beta \in \mathbb{K}$.*
- *The twisting elements of $\mathcal{H}_3$ are given by*
  (1) $F = 1\otimes 1+\beta_1 x\otimes x+\beta_2 y\otimes x$, (2) $F = 1\otimes 1+\beta_1 x\otimes x+\beta_2 x\otimes y$,
  (3) $F = 1\otimes 1+\beta_1 y\otimes x+\beta_2 y\otimes y$, (4) $F = 1\otimes 1+\beta_1 x\otimes y+\beta_2 y\otimes y$,
  with β_1, $\beta_2 \in \mathbb{K}$.
- *The twisting elements of $\mathcal{H}_4$ are given by*
  (1) $F = 1 \otimes 1 + \beta x \otimes x$, (2) $F = 1 \otimes 1 + \beta y \otimes y - 2\beta xy \otimes y$, *with* $\beta \in \mathbb{K}$.

3.1. **Applications.** We discuss in this section an example of applications. We consider the 4-dimensional Hopf superalgebra $\mathcal{H}_2 = \mathbb{K}\langle x, y\rangle/(x^2-x, y^2, xy+yx-y)$ with $\deg(x) = 0$, $\deg(y) = 1$ and such that

$$
\begin{aligned}
\Delta(x) &= 1 \otimes x + x \otimes 1 - 2x \otimes x, & \Delta(y) &= 1 \otimes y + y \otimes 1 - 2x \otimes y - 2y \otimes x, \\
\epsilon(x) &= \epsilon(y) = 0, & S(x) &= x,\ S(y) = y.
\end{aligned}
$$

We consider the 4-dimensional superalgebra (A, μ_A, η_A), defined with respect to a basis $\{v_1^0, v_2^0, v_1^1, v_2^1\}$ by the following relations

$$
\begin{aligned}
&v_2^0 \cdot v_2^0 = v_2^0, v_2^0 \cdot v_1^1 = v_1^1, v_1^1 \cdot v_2^1 = v_2^0, \\
&v_2^1 \cdot v_2^0 = v_2^1, v_2^1 \cdot v_1^1 = v_1^0 - v_2^0, v_2^0 \cdot v_2^1 = v_1^1 \cdot v_2^0 = v_1^1 \cdot v_1^1 = v_2^1 \cdot v_2^1 = 0.
\end{aligned}
$$

We define on A an $\mathcal{H}_2$-supermodule superalgebra structure given by

$$x \triangleright v_2^0 = -\frac{1}{2}v_1^0 + v_2^0, x \triangleright v_1^1 = \frac{1}{2}v_1^1 + v_2^1, x \triangleright v_2^1 = \frac{1}{4}v_1^1 + \frac{1}{2}v_2^1,$$
$$y \triangleright v_2^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$$

We consider $F = 1 \otimes 1 + x \otimes x$ a Drinfeld twist of $\mathcal{H}_2$, its inverse is given by $F^{-1} = 1 \otimes 1 - \frac{1}{2}x \otimes x$.
Since A is a $\mathcal{H}_2$-supermodule superalgebra, according to Theorem 3.4, $A_F = (A, \mu_F, \eta_A)$ is a superalgebra, where the new multiplication μ_F is defined by the following relations
$v_2^0 \cdot v_2^0 = \frac{1}{4}v_1^0 + v_2^0$, $v_2^0 \cdot v_1^1 = \frac{5}{4}v_1^1 - \frac{1}{2}v_2^1$, $v_2^0 \cdot v_2^1 = \frac{1}{8}v_1^1 - \frac{1}{4}v_2^1$, $v_1^1 \cdot v_2^0 = -\frac{1}{4}v_1^1 + \frac{1}{2}v_2^1$, $v_1^1 \cdot v_1^1 = \frac{1}{2}v_1^0$, $v_1^1 \cdot v_2^1 = \frac{1}{4}v_1^0 + v_2^0$, $v_2^1 \cdot v_2^0 = -\frac{1}{8}v_1^1 + \frac{5}{4}v_2^1$, $v_2^1 \cdot v_1^1 = \frac{5}{4}v_1^0 - v_2^0$, $v_2^1 \cdot v_2^1 = \frac{1}{8}v_1^0$.
The superalgebras A and A_F are non-isomorphic, which means that the twisting is non trivial.

By Proposition 3.5, $\mathcal{H}_2^F = (\mathcal{H}_2, \mu, 1, \Delta_F, \varepsilon)$ is a superbialgebra, where the new comultiplication Δ_F is defined by the following relations
$\Delta_F(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x$, $\Delta_F(y) = 1 \otimes y + y \otimes 1 - \frac{3}{2}x \otimes y - \frac{3}{2}y \otimes x - \frac{3}{2}x \otimes xy - \frac{3}{2}xy \otimes x$. Notice that the superbialgebras $\mathcal{H}_2$ and $\mathcal{H}_2^F$ are non-isomorphic.
By Proposition 3.6, A_F is a B_F-supermodule superalgebra, where the new supermodule superalgebra structure is defined by

$$x \triangleright v_1^1 = v_1^1,\ x \triangleright v_2^1 = v_2^1,\ y \triangleright v_2^0 = xy \triangleright v_2^0 = \lambda_1 v_1^1 + \lambda_2 v_2^1,$$
$$y \triangleright v_1^1 = -2\lambda_2 v_1^0,\ y \triangleright v_2^1 = 2\lambda_1 v_1^0,$$
$$x \triangleright v_2^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$$

4. **2-Cocycle Deformations**

In this section, we study 2-cocycle deformations in the $\mathbb{Z}_2$-grading case. As in the non-graded case, 2-cocycles provide a way of modifying the multiplication of a superbialgebra in order to produce another superbialgebra [8].
Let B be a superbialgebra over $\mathbb{K}$ and let $\sigma : B \otimes B \to \mathbb{K}$ be a morphism. We say that σ is convolution invertible if and only if there is a morphism $\sigma^{-1} : B \otimes B \to \mathbb{K}$, called the inverse of σ, which satisfies $(\sigma * \sigma^{-1})(a \otimes b) = (\sigma^{-1} * \sigma)(a \otimes b) = (-1)^{|a_{(2)}||b_{(1)}|}\sigma(a_{(1)} \otimes b_{(1)})\sigma^{-1}(a_{(2)} \otimes b_{(2)}) = \varepsilon(a)\varepsilon(b)$ for all $a, b \in B$.
The two notions of "2-cocycle" and of "twisting" are dual to each other [17].

Definition 4.1. *Let $(B, \mu, 1_B, \Delta, \varepsilon)$ be a superbialgebra over $\mathbb{K}$. A convolution invertible morphism $\sigma : B \otimes B \to \mathbb{K}$ is called unital (or normalized) 2-cocycle for B when, for all $a, b, c \in B$, the following two conditions are satisfied*

$$\sigma(a_{(1)} \otimes b_{(1)})\sigma(a_{(2)}b_{(2)} \otimes c) = \sigma(b_{(1)} \otimes c_{(1)})\sigma(a \otimes b_{(2)}c_{(2)}), \tag{4.1}$$
$$\sigma(a \otimes 1) = \varepsilon(a) = \sigma(1 \otimes a). \tag{4.2}$$

These two conditions are equivalents to

$(\varepsilon \otimes \sigma) * \sigma(id_B \otimes \mu) = (\sigma \otimes \varepsilon) * \sigma(\mu \otimes id_B)$ *and* $\sigma(id_B \otimes 1) = \varepsilon = \sigma(1 \otimes id_B)$.

In the sequel, we will simply call σ a 2-cocycle instead of unital 2-cocycle if no confusion arises.
Using a 2-cocycle σ, it is possible to define a new superalgebra structure on B by deforming the multiplication, while the new multiplication is defined via a formula which only depends on the initial multiplication and on a 2-cocycle σ of the superbialgebra B.

Proposition 4.2. *Let $(B, \mu, 1_B, \Delta, \varepsilon)$ be a superbialgebra over $\mathbb{K}$ and suppose that σ is a 2-cocycle for B. Then $B_\sigma = (B, \mu_\sigma, 1, \Delta, \varepsilon)$ is a superbialgebra over $\mathbb{K}$, where the multiplication μ_σ is defined as*

$$\mu_\sigma = \sigma * \mu * \sigma^{-1}.$$

Proof. We show that $(B, \mu_\sigma, 1_B)$ is a superalgebra over $\mathbb{K}$.
The associativity of the multiplication μ_σ follows from the 2-cocycle condition (4.1) and the evenly of σ. Condition (4.2) implies that $\mu_\sigma(x \otimes 1) = x = \mu_\sigma(1 \otimes x)$ [8]. Moreover the multiplication μ_σ is compatible with Δ and the unit 1_B is a supercoalgebra morphism. Hence B_σ is a superbialgebra. □

Now we construct all 2-cocycles of low-dimensional Hopf superalgebras obtained in [1].

Proposition 4.3 (**2-cocycles of 2-dimensional Hopf superalgebra**). *The 2-cocycles of the 2-dimensional Hopf superalgebra $\mathcal{H}$ (see (1.7)) are given by $\sigma(x \otimes x) = \lambda$, and its inverse $\sigma^{-1}(x \otimes x) = -\lambda$ with $\lambda \in \mathbb{K}$.*

Proposition 4.4 (**2-cocycles of 4-dimensional Hopf superalgebras**).

- *The 2-cocycles of $\mathcal{H}_0$ (see (1.9)) are given by*
  $\sigma(x^2 \otimes x^2) = 1, \sigma(y \otimes y) = \lambda, \sigma(x \otimes x) = \sigma(x \otimes x^2) = \sigma(x^2 \otimes x) = 0,$
  and the inverse by
  $\sigma^{-1}(x^2 \otimes x^2) = 1, \sigma^{-1}(y \otimes y) = \frac{1}{\lambda}, \sigma^{-1}(x \otimes x) = \sigma^{-1}(x \otimes x^2) = \sigma^{-1}(x^2 \otimes x) = 0,$ *with* $\lambda \in \mathbb{K}^*$.
- *The 2-cocycles of $\mathcal{H}_1$ are given by*
  $\sigma(x \otimes x) = \lambda_1, \sigma(y \otimes y) = \lambda_2, \sigma(xy \otimes xy) = \lambda_1\lambda_2, \sigma(y \otimes xy) = \sigma(xy \otimes y) = 0,$
  and the inverse by $\sigma^{-1}(x \otimes x) = \frac{-\lambda_1}{1+4\lambda_1}, \sigma^{-1}(y \otimes y) = -\lambda_2, \sigma^{-1}(xy \otimes xy) = \frac{\lambda_1\lambda_2}{1+4\lambda_1},$
  $\sigma^{-1}(y \otimes xy) = \sigma^{-1}(xy \otimes y) = 0,$ *with* $\lambda_1 \in \mathbb{K} - \{-\frac{1}{4}\}$ *and* $\lambda_2 \in \mathbb{K}$.
- *The 2-cocycles of $\mathcal{H}_2$ are given by*
  1. $\sigma(x \otimes x) = \lambda_1, \sigma(y \otimes y) = \lambda_2, \sigma(y \otimes xy) = \frac{(1+2\lambda_1)\lambda_2}{1+4\lambda_1}, \sigma(xy \otimes y) = \frac{2\lambda_1\lambda_2}{1+4\lambda_1},$
     $\sigma(xy \otimes xy) = \frac{\lambda_1\lambda_2}{1+4\lambda_1},$ *and its inverse* $\sigma^{-1}(x \otimes x) = \frac{-\lambda_1}{1+4\lambda_1}, \sigma^{-1}(y \otimes y) = \frac{-\lambda_2}{(1+4\lambda_1)^2},$
     $\sigma^{-1}(y \otimes xy) = \frac{-(1+2\lambda_1)\lambda_2}{(1+4\lambda_1)^2}, \sigma^{-1}(xy \otimes y) = \frac{2\lambda_1\lambda_2}{(1+4\lambda_1)^2}, \sigma^{-1}(xy \otimes xy) = \frac{\lambda_1\lambda_2}{(1+4\lambda_1)^2},$
  2. $\sigma(y \otimes y) = \sigma(y \otimes xy) = \lambda, \sigma(x \otimes x) = \sigma(xy \otimes y) = \sigma(xy \otimes xy) = 0,$
     and its inverse $\sigma^{-1}(y \otimes y) = \sigma^{-1}(y \otimes xy) = -\lambda, \sigma^{-1}(x \otimes x) = \sigma^{-1}(xy \otimes y) = \sigma^{-1}(xy \otimes xy) = 0,$ *with* $\lambda_1 \in \mathbb{K} - \{-\frac{1}{4}\}$ *and* $\lambda_2 \in \mathbb{K}$.

- *The 2-cocycles of $\mathcal{H}_3$ are given by* $\sigma(x\otimes x)=\lambda_1, \sigma(x\otimes y)=\lambda_2, \sigma(y\otimes x)=\lambda_3, \sigma(y\otimes y)=\lambda_4, \sigma(xy\otimes xy)=\lambda_2\lambda_3-\lambda_1\lambda_4$*, and the inverse by* $\sigma^{-1}(x\otimes x)=-\lambda_1, \sigma^{-1}(x\otimes y)=-\lambda_2, \sigma^{-1}(y\otimes x)=-\lambda_3, \sigma^{-1}(y\otimes y)=-\lambda_4, \sigma^{-1}(xy\otimes xy)=\lambda_2\lambda_3-\lambda_1\lambda_4$*, with* $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in\mathbb{K}$.
- *The 2-cocycles of $\mathcal{H}_4$ are given by* $\sigma(x\otimes x)=\lambda_1, \sigma(xy\otimes xy)=\lambda_2, \sigma(y\otimes y)=\frac{(1+4\lambda_1)\lambda_2}{\lambda_1}, \sigma(y\otimes xy)=\sigma(xy\otimes y)=2\lambda_2$*, and the inverse by* $\sigma^{-1}(x\otimes x)=\frac{-\lambda_1}{1+4\lambda_1}, \sigma^{-1}(xy\otimes xy)=\frac{\lambda_2}{1+4\lambda_1}, \sigma^{-1}(y\otimes y)=-\frac{\lambda_2}{\lambda_1}, \sigma^{-1}(y\otimes xy)=\sigma(xy\otimes y)=0$*, with* $\lambda_1\in\mathbb{K}^*-\{-\frac{1}{4}\}$ *and* $\lambda_2\in\mathbb{K}$.

4.1. **Applications.** We show in the following an example of deformation of a multiplication using a 2-cocycle. We consider the 4-dimensional Hopf superalgebra $\mathcal{H}_4=\mathbb{K}[x,y]/(x^2-x,y^2)$ with $\deg(x)=0$, $\deg(y)=1$ and such that

$$\begin{aligned}\Delta(x)&=1\otimes x+x\otimes 1-2x\otimes x, & \Delta(y)&=1\otimes y+y\otimes 1-2x\otimes y,\\ \epsilon(x)&=\epsilon(y)=0, & S(x)&=x,\ S(y)=-y+2xy,\end{aligned}$$

We set $\lambda_1=\lambda_2=1$ for the 2-cocycle of $\mathcal{H}_4$ in Proposition 4.4. By Proposition 4.2, $\mathcal{H}_4^\sigma=(\mathcal{H}_4,\mu_\sigma,1,\Delta,\varepsilon)$ is a superbialgebra, where the new multiplication μ_σ is defined by the following relations

$$\begin{aligned}&\mu_\sigma(x\otimes x)=x, \mu_\sigma(x\otimes y)=\mu_\sigma(y\otimes x)=-2y+5xy,\\ &\mu_\sigma(x\otimes xy)=\mu_\sigma(xy\otimes x)=-\frac{6}{5}y+3xy,\\ &\mu_\sigma(y\otimes y)=\mu_\sigma(y\otimes xy)=\mu_\sigma(xy\otimes y)=\mu_\sigma(xy\otimes xy)=0.\end{aligned}$$

It turns out that $(\mathcal{H}_4,\mu,1)$ and $(\mathcal{H}_4,\mu_\sigma,1)$ are two isomorphic superalgebras.

Appendix A

We provide in the following supermodule, supercomodule and supermodule superalgebra structures over Hopf superalgebras $\mathcal{H}_1$, $\mathcal{H}_2, \mathcal{H}_3$ and $\mathcal{H}_4$ in dimension m with $m\le 3$. In the sequel, α_i, γ_i and λ_j for $i\in\{1,\cdots,4\}, j\in\{1,2\}$ denote parameters in $\mathbb{K}$. Note that all the obtained structures are non-isomorphic.
The superalgebra structure of $\mathcal{H}_1$ and $\mathcal{H}_4$ are defined as a quotient

$$\mathbb{K}[x,y]/(x^2-x,y^2),\ \text{with}\ \deg(x)=0,\deg(y)=1.$$

Consider a basis $\{1,x,y,xy\}$ and relations $x^2=x,\ y^2=0$, the Hopf superalgebra $\mathcal{H}_1$ is defined by the following comultiplication, counit and antipode

$$\begin{aligned}\Delta(x)&=1\otimes x+x\otimes 1-2x\otimes x, & \Delta(y)&=1\otimes y+y\otimes 1,\\ \epsilon(x)&=\epsilon(y)=0, & S(x)&=x,\ S(y)=-y.\end{aligned}$$

and the Hopf superalgebra $\mathcal{H}_4$ is defined by the following comultiplication, counit and antipode

$$\begin{aligned}\Delta(x)&=1\otimes x+x\otimes 1-2x\otimes x, & \Delta(y)&=1\otimes y+y\otimes 1-2x\otimes y,\\ \epsilon(x)&=\epsilon(y)=0, & S(x)&=x,\ S(y)=-y+2xy.\end{aligned}$$

The superalgebra structure of $\mathcal{H}_2$ is defined as a quotient

$$\mathbb{K}\langle x, y\rangle/(x^2 - x, y^2, xy + yx - y), \text{ with } \deg(x) = 0, \deg(y) = 1.$$

Consider a basis $\{1, x, y, xy\}$ and relations $x^2 = x,\ y^2 = 0,\ yx = y - xy$, the Hopf superalgebra $\mathcal{H}_2$ is defined by the following comultiplication, counit and antipode

$$\Delta(x) = 1 \otimes x + x \otimes 1 - 2x \otimes x, \quad \Delta(y) = 1 \otimes y + y \otimes 1 - 2x \otimes y - 2y \otimes x,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = x,\ S(y) = y.$$

The superalgebra structure of $\mathcal{H}_3$ is defined as a quotient

$$\mathbb{K}\langle x, y\rangle/(x^2, y^2, xy + yx), \text{ with } \deg(x) = \deg(y) = 1.$$

Consider a basis $\{1, xy, x, y\}$ and relations $x^2 = 0,\ y^2 = 0,\ yx = -xy$, the Hopf superalgebra $\mathcal{H}_3$ is defined by the following comultiplication, counit and antipode

$$\Delta(x) = 1 \otimes x + x \otimes 1, \quad \Delta(y) = 1 \otimes y + y \otimes 1,$$
$$\epsilon(x) = \epsilon(y) = 0, \qquad S(x) = -x,\ S(y) = -y.$$

Most of the results are obtained by using the computer algebra system Mathematica. The program we have used is available.

A.1. **Low dimensional supermodules over $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$.** Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0, 1\}, p \in \{1, \cdots, m_q\}$.

Proposition A.1. *The supermodule structures over $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_4$ are defined in the different cases as follows.*

- *Case* $m_0 = 1,\ m_1 = 0.$ $x \triangleright v_1^0 = v_1^0.$
- *Case* $m_0 = 0,\ m_1 = 1.$ $x \triangleright v_1^1 = v_1^1.$
- *Case* $m_0 = 2,\ m_1 = 0.$ (1) $x \triangleright v_1^0 = \alpha_1 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} + (1 - \alpha_1)v_2^0, (\alpha_2 \neq 0)$
(2) $x \triangleright v_1^0 = 0, x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0,$ (3) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0,$ (4) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0.$
- *Case* $m_0 = 0,\ m_1 = 2.$ (1) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1,$ (2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1.$
- *Case* $m_0 = 3,\ m_1 = 0.$
  (1) $x \triangleright v_2^0 = \alpha_1\alpha_2 v_1^0 + \alpha_1 v_3^0, x \triangleright v_3^0 = \alpha_2 v_1^0 + v_3^0, x \triangleright v_1^0 = 0,$
  (2) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \alpha_1 v_1^0 + \alpha_2 v_2^0,$
  (3) $x \triangleright v_1^0 = v_1^0 - \alpha_1\alpha_2 v_2^0 + \alpha_1 v_3^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \alpha_2 v_2^0,$
  (4) $x \triangleright v_1^0 = \alpha_1 v_2^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \alpha_2 v_2^0,$
  (5) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_3^0 = \alpha_2 v_1^0 + v_3^0, x \triangleright v_1^0 = 0,$
  (6) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, x \triangleright v_3^0 = \alpha_2 v_1^0,$
  (7) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = v_3^0.$
- *Case* $m_0 = 0,\ m_1 = 3.$
  (1) $x \triangleright v_1^1 = \alpha_1 v_2^1, x \triangleright v_2^1 = v_2^1, x \triangleright v_3^1 = \alpha_2 v_2^1,$
  (2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, x \triangleright v_3^1 = \alpha_1 v_1^1 + \alpha_2 v_2^1,$
  (3) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, x \triangleright v_3^1 = v_3^1.$

Proposition A.2. *The supermodule structures over $\mathcal{H}_1$ and $\mathcal{H}_4$ are defined in the different cases as follows.*

- *Case* $m_0 = 1,\ m_1 = 1$.

(1) $y \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(2) $y \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(3) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(4) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = 0,$
(5) $x \triangleright v_1^1 = v_1^1, x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(6) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0.$

- *Case* $m_0 = 1,\ m_1 = 2$.

(1) $x \triangleright v_1^1 = \alpha_1 v_2^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^0 = \alpha_2 v_1^1 - \alpha_1\alpha_2 v_2^1, x \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_1^2 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_2^1 = \alpha_2 v_1^0, x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(3) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_2 v_1^0, y \triangleright v_2^1 = xy \triangleright v_2^1 = \alpha_1\alpha_2 v_1^0, y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(4) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1 + \alpha_1 v_2^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_2 v_1^1 + \alpha_1\alpha_2 v_2^1, x \triangleright v_2^1 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(5) $y \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_2^1 = \alpha_2 v_1^0, x \triangleright v_1^0 = x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^0 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(6) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_2^1 = xy \triangleright v_2^1 = \alpha_2 v_1^0,$
$y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(7) $y \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1, x \triangleright v_1^0 = x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(8) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1, y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(9) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(10) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$

- *Case* $m_0 = 2,\ m_1 = 1$.

(1) $x \triangleright v_1^0 = \alpha_1 v_2^0, x \triangleright v_2^0 = v_2^0, y \triangleright v_1^1 = \alpha_2 v_1^0 - \alpha_1\alpha_2 v_2^0, x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(2) $x \triangleright v_1^0 = (1-\alpha_1) v_1^0 + \alpha_2 v_2^0, y \triangleright v_1^0 = \alpha_3 v_1^1, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} v_1^0 + \alpha_1 v_2^0, y \triangleright v_2^0 = \frac{\alpha_1\alpha_3 - \alpha_3}{\alpha_2} v_1^1, x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0, (\alpha_2 \neq 0),$
(3) $x \triangleright v_1^0 = \alpha_1 v_2^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_2 v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \frac{\alpha_2}{\alpha_1} v_1^1,$
$y \triangleright v_1^1 = xy \triangleright v_1^1 = 0, (\alpha_1 \neq 0),$
(4) $x \triangleright v_1^0 = v_1^0 + \alpha_1 v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_2 v_1^0 + \alpha_1\alpha_2 v_2^0$ $x \triangleright v_2^0 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(5) $y \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_2^0 = \alpha_2 v_1^1, x \triangleright v_1^0 = x \triangleright v_2^0 = x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(6) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \alpha_2 v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = 0,$

(7) $y \triangleright v_1^1 = \alpha_1 v_1^0 + \alpha_2 v_2^0, x \triangleright v_1^0 = x \triangleright v_2^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(8) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0 + \alpha_2 v_2^0, y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(9) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_2 v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \alpha_1\alpha_2 v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = 0,$
(10) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, y \triangleright v_2^0 = \alpha_2 v_1^1, x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(11) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \alpha_2 v_1^1, x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(12) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, y \triangleright v_1^0 = \alpha_2 v_1^1, y \triangleright v_2^0 = -\alpha_1\alpha_2 v_1^1, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(13) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_2^0 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(14) $x \triangleright v_1^0 = v_1^0, y \triangleright v_1^1 = \alpha_1 v_2^0, x \triangleright v_2^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(15) $x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_2^0, x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(16) $x \triangleright v_2^0 = v_2^0, y \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(17) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(18) $x \triangleright v_1^1 = v_1^1, x \triangleright v_1^0 = x \triangleright v_2^0 = y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0.$

Proposition A.3. *The supermodule structures over $\mathcal{H}_2$ are defined in the different cases as follows.*

- *Case* $m_0 = 1,\ m_1 = 1.$

(1) $x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_1^0 = y \triangleright v_1^0 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(2) $x \triangleright v_1^0 = v_1^0, y \triangleright v_1^1 = \alpha_1 v_1^0, xy \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_1^1 = y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(3) $x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = \alpha_1 v_1^1, xy \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^1 = 0,$
(4) $x \triangleright v_1^0 = v_1^0, y \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(5) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0.$

- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^1 = \alpha_1 v_1^1 + v_2^1, y \triangleright v_1^1 = y \triangleright v_1^1 = \alpha_2 v_1^1, y \triangleright v_2^1 = xy \triangleright v_2^1 = -\alpha_1\alpha_2 v_1^0,$
$x \triangleright v_1^1 = y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_1^0 = \alpha_2 v_1^0, y \triangleright v_2^1 = \alpha_1\alpha_2 v_1^0,$
$x \triangleright v_1^0 = y \triangleright v_1^0 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(3) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_2 v_1^1,$
$x \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(4) $x \triangleright v_1^0 = v_1^0, x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \alpha_1 v_1^1, y \triangleright v_1^0 = -\alpha_1\alpha_2 v_1^1 + \alpha_2 v_2^1,$
$y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(5) $x \triangleright v_1^0 = v_1^0, y \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1,$
$x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$

(6) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1,$
$x \triangleright v_1^0 = y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(7) $x \triangleright v_1^0 = v_1^0, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_2^1 = xy \triangleright v_2^1 = \alpha_2 v_1^0,$
$x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^0 = xy \triangleright v_1^0 = 0,$
(8) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_2^1 = \alpha_2 v_1^0,$
$x \triangleright v_1^0 = y \triangleright v_1^0 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$

- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $x \triangleright v_1^0 = (1-\alpha_1)\, v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} v_1^0 + \alpha_1 v_2^0,$
$y \triangleright v_1^1 = xy \triangleright v_1^1 = \frac{\alpha_3 - \alpha_1\alpha_3}{\alpha_2} v_1^0 + \alpha_3 v_2^0, x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 =$
$xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0, (\alpha_2 \neq 0)$
(2) $x \triangleright v_1^0 = (1-\alpha_1)\, v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} v_1^0 + \alpha_1 v_2^0, x \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^1 = \frac{\alpha_3 - \alpha_1\alpha_3}{\alpha_2} v_1^0 + \alpha_3 v_2^0, y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 =$
$xy \triangleright v_1^1 = 0, (\alpha_2 \neq 0)$
(3) $x \triangleright v_1^0 = (1-\alpha_1)\, v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} v_1^0 + \alpha_1 v_2^0, x \triangleright v_1^1 = v_1^1,$
$y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_3 v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \frac{\alpha_1\alpha_3 - \alpha_3}{\alpha_2} v_1^1, y \triangleright v_1^1 =$
$xy \triangleright v_1^1 = 0, (\alpha_2 \neq 0)$
(4) $x \triangleright v_1^0 = (1-\alpha_1) v_1^0 + \alpha_2 v_2^0, x \triangleright v_2^0 = \frac{\alpha_1 - \alpha_1^2}{\alpha_2} v_1^0 + \alpha_1 v_2^0, y \triangleright v_1^0 =$
$\alpha_3 v_1^1, y \triangleright v_2^0 = \frac{\alpha_1\alpha_3}{\alpha_2} v_1^1,$
$x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0, (\alpha_2 \neq 0)$
(5) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1 v_1^0 + \alpha_2 v_2^0$
$x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(6) $x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \alpha_2 v_1^1$
$x \triangleright v_1^0 = x \triangleright v_2^0 = y \triangleright v_1^1 = xy \triangleright v_1^1 = 0,$
(7) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, y \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_2^0 = \alpha_2 v_1^1$
$x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(8) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_2 v_1^0,$
$x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(9) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = -\alpha_1\alpha_2 v_1^0 + \alpha_1 v_2^0,$
$y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(10) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = \alpha_2 v_1^1,$
$y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_1^1 = 0,$
(11) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = \alpha_1 v_1^0, y \triangleright v_1^0 = \alpha_2 v_1^1, y \triangleright v_2^0 = \alpha_1\alpha_2 v_1^1,$
$x \triangleright v_1^1 = y \triangleright v_1^1 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(12) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, y \triangleright v_1^1 = xy \triangleright v_1^1 = \alpha_1\alpha_2 v_1^0 + \alpha_2 v_2^0,$
$x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0,$
(13) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = \alpha_2 v_1^0,$
$x \triangleright v_1^0 = y \triangleright v_1^0 = y \triangleright v_2^0 = xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(14) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, y \triangleright v_2^0 = \alpha_2 v_1^1, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_1^1 =$
$xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(15) $x \triangleright v_2^0 = \alpha_1 v_1^0 + v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = xy \triangleright v_1^0 = \alpha_2 v_1^1, y \triangleright v_2^0 =$
$xy \triangleright v_2^0 = -\alpha_1\alpha_2 v_1^1, x \triangleright v_1^0 = x \triangleright v_1^1 = y \triangleright v_1^0 = y \triangleright v_1^1 = xy \triangleright v_1^0 =$
$xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(16) $x \triangleright v_1^0 = v_1^0, x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_1^0 = y \triangleright v_2^0 = y \triangleright v_1^1 =$
$xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0.$

Proposition A.4. *The supermodule structures over $\mathcal{H}_3$ are defined in the different cases as follows.*
- *Case* $m_0 = 1,\ m_1 = 0.$ $xy \triangleright v_1^0 = 0.$
- *Case* $m_0 = 0,\ m_1 = 1.$ $xy \triangleright v_1^1 = 0.$
- *Case* $m_0 = 2,\ m_1 = 0.$ $xy \triangleright v_1^0 = xy \triangleright v_2^0 = 0.$
- *Case* $m_0 = 0,\ m_1 = 2.$ $xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$
- *Case* $m_0 = 1,\ m_1 = 1.$
  (1) $x \triangleright v_1^1 = \alpha_1 v_1^0, y \triangleright v_1^1 = \alpha_2 v_1^0, xy \triangleright v_1^0 = xy \triangleright v_1^1 = x \triangleright v_1^0 = y \triangleright v_1^0 = 0,$
  (2) $x \triangleright v_1^0 = \alpha_1 v_1^1, y \triangleright v_1^0 = \alpha_2 v_1^1, xy \triangleright v_1^0 = xy \triangleright v_1^1 = x \triangleright v_1^1 = y \triangleright v_1^1 = 0.$

Proposition A.5. *The supermodule structures over $\mathcal{H}_3$ are defined in the different cases as follows.*
- *Case* $m_0 = 3,\ m_1 = 0.$ $xy \triangleright v_1^0 = xy \triangleright v_2^0 = xy \triangleright v_3^0 = 0.$
- *Case* $m_0 = 0,\ m_1 = 3.$ $xy \triangleright v_1^1 = xy \triangleright v_2^1 = xy \triangleright v_3^1 = 0.$
- *Case* $m_0 = 1,\ m_1 = 2.$
  (1) $x \triangleright v_1^1 = \alpha_1 v_1^0, x \triangleright v_2^1 = \alpha_2 v_1^0, y \triangleright v_1^1 = \alpha_3 v_1^0, y \triangleright v_2^1 = \alpha_4 v_1^0,$
  $xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = x \triangleright v_1^0 = y \triangleright v_1^0 = 0,$
  (2) $x \triangleright v_1^0 = \alpha_1 v_1^1 + \alpha_2 v_2^1, y \triangleright v_1^0 = \alpha_3 v_1^1 + \alpha_4 v_2^1,$
  $xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0.$
- *Case* $m_0 = 2,\ m_1 = 1.$
  (1) $x \triangleright v_1^0 = \alpha_1 v_1^1, x \triangleright v_2^0 = \alpha_2 v_1^1, y \triangleright v_1^0 = \alpha_3 v_1^1, y \triangleright v_2^0 = \alpha_4 v_1^1,$
  $xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = x \triangleright v_1^0 = y \triangleright v_1^0 = 0,$
  (2) $x \triangleright v_1^1 = \alpha_1 v_1^0 + \alpha_2 v_2^0, y \triangleright v_1^1 = \alpha_3 v_1^0 + \alpha_4 v_2^0,$
  $xy \triangleright v_1^0 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = x \triangleright v_1^1 = x \triangleright v_2^1 = y \triangleright v_1^1 = y \triangleright v_2^1 = 0.$

A.2. **Low dimensional supercomodules over $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$.** Let V be a superspace of dimension $m = m_0 + m_1$, where m_0 is the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of V, where $q \in \{0, 1\}, p \in \{1, \cdots, m_q\}$.

Proposition A.6. *The supercomodule structures over $\mathcal{H}_1$, $\mathcal{H}_2$ and $\mathcal{H}_4$ are defined in the different cases as follows.*
- *Case* $m_0 = 1,\ m_1 = 0.$ (1) $\rho(v_1^0) = 1 \otimes v_1^0,$ (2) $\rho(v_1^0) = (1 - 2x) \otimes v_1^0.$
- *Case* $m_0 = 0,\ m_1 = 1.$ (1) $\rho(v_1^1) = 1 \otimes v_1^1$, (2) $\rho(v_1^1) = (1 - 2x) \otimes v_1^1.$
- *Case* $m_0 = 2,\ m_1 = 0.$
  (1) $\rho(v_1^0) = (1 + \gamma_1 x) \otimes v_1^0 + \gamma_2 x \otimes v_2^0, \rho(v_2^0) = -\frac{2\gamma_1 + \gamma_1^2}{\gamma_2} x \otimes v_1^0 + (1 - (2 + \gamma_1)\, x) \otimes v_2^0, (\gamma_2 \neq 0)$
  (2) $\rho(v_1^0) = (1 - 2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0,$
  (3) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + (1 - 2x) \otimes v_2^0,$
  (4) $\rho(v_1^0) = (1 - 2x) \otimes v_1^0, \rho(v_2^0) = (1 - 2x) \otimes v_2^0,$
  (5) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0.$
- *Case* $m_0 = 0,\ m_1 = 2.$
  (1) $\rho(v_1^1) = (1 - 2x) \otimes v_1^1, \rho(v_2^1) = \gamma_1 x \otimes v_1^1 + 1 \otimes v_2^1,$
  (2) $\rho(v_1^1) = (1 - 2x) \otimes v_1^1, \rho(v_2^1) = (1 - 2x) \otimes v_2^1,$
  (3) $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1.$
- *Case* $m_0 = 3,\ m_1 = 0.$
  (1) $\rho(v_1^0) = (1 - 2x) \otimes v_1^0, \rho(v_2^0) = (1 - 2x) \otimes v_2^0,$
  $\rho(v_3^0) = \gamma_1 x \otimes v_1^0 + \gamma_2 x \otimes v_2^0 + 1 \otimes v_3^0,$

(2) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_2^0)=1\otimes v_2^0, \rho(v_3^0)=\gamma_1 x\otimes v_1^0+\gamma_2 x\otimes v_2^0+(1-2x)\otimes v_3^0$,
(3) $\rho(v_1^0)=1\otimes v_1^0+\gamma_1 x\otimes v_2^0, \rho(v_2^0)=(1-2x)\otimes v_2^0, \rho(v_3^0)=\gamma_2 x\otimes v_2^0+1\otimes v_3^0$,
(4) $\rho(v_1^0)=(1-2x)\otimes v_1^0+\gamma_1 x\otimes v_2^0, \rho(v_2^0)=1\otimes v_2^0, \rho(v_3^0)=\gamma_2 x\otimes v_2^0+(1-2x)\otimes v_3^0$,
(5) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_2^0)=\gamma_1 x\otimes v_1^0+1\otimes v_2^0, \rho(v_3^0)=\gamma_2 x\otimes v_1^0+1\otimes v_3^0$,
(6) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_2^0)=\gamma_1 x\otimes v_1^0+(1-2x)\otimes v_2^0, \rho(v_3^0)=\gamma_2 x\otimes v_1^0+(1-2x)\otimes v_3^0$,
(7) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_2^0)=(1-2x)\otimes v_2^0, \rho(v_3^0)=(1-2x)\otimes v_3^0$,
(8) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_2^0)=1\otimes v_2^0, \rho(v_3^0)=1\otimes v_3^0$.

- *Case* $m_0=0,\ m_1=3$.

(1) $\rho(v_1^1)=(1-2x)\otimes v_1^1, \rho(v_2^1)=(1-2x)\otimes v_2^1$,
$\rho(v_3^1)=\gamma_1 x\otimes v_1^1+\gamma_2 x\otimes v_2^1+1\otimes v_3^1$,
(2) $\rho(v_1^1)=(1-2x)\otimes v_1^1, \rho(v_2^1)=(1-2x)\otimes v_2^1, \rho(v_3^1)=(1-2x)\otimes v_3^1$,
(3) $\rho(v_1^1)=1\otimes v_1^1, \rho(v_2^1)=1\otimes v_2^1, \rho(v_3^1)=1\otimes v_3^1$.

Proposition A.7. *The supercomodule structures over $\mathcal{H}_1$ are defined in the different cases as follows.*

- *Case* $m_0=1,\ m_1=1$.

(1) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_1^1)=(\gamma_1 y-2\gamma_1 xy)\otimes v_1^0+(1-2x)\otimes v_1^1$,
(2) $\rho(v_1^0)=(1-2x)\otimes v_1^0+(\gamma_1 y-2\gamma_1 xy)\otimes v_1^1, \rho(v_1^1)=(1-2x)\otimes v_1^1$,
(3) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_1^1)=\gamma_1 y\otimes v_1^0+1\otimes v_1^1$,
(4) $\rho(v_1^0)=1\otimes v_1^0+\gamma_1 y\otimes v_1^1, \rho(v_1^1)=1\otimes v_1^1$,
(5) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_1^1)=1\otimes v_1^1$,
(6) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_1^1)=(1-2x)\otimes v_1^1$.

- *Case* $m_0=1,\ m_1=2$.

(1) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_1^1)=\big(\gamma_1(-\frac{1}{2}y+xy)\big)\otimes v_1^0+(1-2x)\otimes v_1^1$,
$\rho(v_2^1)=\big(\gamma_2(-\frac{1}{2}y+xy)\big)\otimes v_1^0+(1-2x)\otimes v_1^1$,
(2) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_1^1)=\gamma_1 y\otimes v_1^0+1\otimes v_1^1, \rho(v_2^1)=\gamma_2 y\otimes v_1^0+1\otimes v_2^1$,
(3) $\rho(v_1^0)=(1-2x)\otimes v_1^0+\big(\gamma_1(-\frac{1}{2}y+xy)\big)\otimes v_1^1+\big(\gamma_2(-\frac{1}{2}y+xy)\big)\otimes v_2^1$,
$\rho(v_1^1)=(1-2x)\otimes v_1^1, \rho(v_2^1)=(1-2x)\otimes v_2^1$,
(4) $\rho(v_1^0)=1\otimes v_1^0+\gamma_1 y\otimes v_1^1+\gamma_2 y\otimes v_2^1, \rho(v_1^1)=1\otimes v_1^1, \rho(v_2^1)=1\otimes v_2^1$,
(5) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_1^1)=1\otimes v_1^1, \rho(v_2^1)=\big(\gamma_1(-\frac{1}{2}y+xy)\big)\otimes v_1^0+\gamma_2 x\otimes v_1^1+(1-2x)\otimes v_2^1$,
(6) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_1^1)=\gamma_1 y\otimes v_1^0+1\otimes v_1^1, \rho(v_2^1)=\frac{1}{2}\gamma_1\gamma_2 y\otimes v_1^0+\gamma_2 x\otimes v_1^1+(1-2x)\otimes v_2^1$,
(7) $\rho(v_1^0)=1\otimes v_1^0+\gamma_1 y\otimes v_1^1+\frac{1}{2}\gamma_1\gamma_2 y\otimes v_2^1, \rho(v_1^1)=1\otimes v_1^1+\gamma_2 x\otimes v_2^1, \rho(v_2^1)=(1-2x)\otimes v_2^1$,
(8) $\rho(v_1^0)=(1-2x)\otimes v_1^0+\big(\gamma_1(-\frac{1}{2}y+xy)\big)\otimes v_2^1, \rho(v_1^1)=1\otimes v_1^1+\gamma_2 x\otimes v_2^1, \rho(v_2^1)=(1-2x)\otimes v_2^1$,
(9) $\rho(v_1^0)=1\otimes v_1^0, \rho(v_1^1)=(1-2x)\otimes v_1^1, \rho(v_2^1)=(1-2x)\otimes v_2^1$,
(10) $\rho(v_1^0)=(1-2x)\otimes v_1^0, \rho(v_1^1)=1\otimes v_1^1, \rho(v_2^1)=1\otimes v_2^1$.

- *Case* $m_0=2,\ m_1=1$.

(1) $\rho(v_1^0)=1\otimes v_1^0+\gamma_1 x\otimes v_2^0, \rho(v_2^0)=(1-2x)\otimes v_2^0, \rho(v_1^1)=\gamma_2 y\otimes v_1^0+\frac{1}{2}\gamma_1\gamma_2 y\otimes v_2^0+1\otimes v_1^1$,

(2) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_2^0 + \gamma_2(-\frac{1}{2}y + xy) \otimes v_2^0, \rho(v_2^0) = (1-2x) \otimes v_2^0 + \frac{\gamma_2(y-2xy)}{\gamma_1} \otimes v_1^1$,
$\rho(v_1^1) = (1-2x) \otimes v_1^1, (\gamma_1 \neq 0)$
(3) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_2^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = \gamma_2(-\frac{1}{2}y + xy) \otimes v_2^0 + (1-2x) \otimes v_1^1$,
(4) $\rho(v_1^0) = (1-2x) \otimes v_2^0 + \gamma_1(-\frac{1}{2}y + xy) \otimes v_1^1, \rho(v_2^0) = (1-2x) \otimes v_2^0 + \gamma_2(-\frac{1}{2}y + xy) \otimes v_1^1$,
$\rho(v_1^1) = (1-2x) \otimes v_1^1$,
(5) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0 + \gamma_2 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,
(6) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0$,
$\rho(v_1^1) = \gamma_1(-\frac{1}{2}y + xy) \otimes v_1^0 + \gamma_2(-\frac{1}{2}y + xy) \otimes v_2^0 + (1-2x) \otimes v_1^1$,
(7) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + \gamma_2 y \otimes v_2^0 + 1 \otimes v_1^1$,
(8) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_2^0) = \gamma_2 x \otimes v_1^0 + (1-2x) \otimes v_2^0 + \frac{1}{2}\gamma_1\gamma_2 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,
(9) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + (1-2x) \otimes v_2^0 + \gamma_2(-\frac{1}{2}y + xy) \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1$,
(10) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0 + \gamma_2 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,
(11) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1(-\frac{1}{2}y + xy) \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$,
(12) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + 1 \otimes v_1^1$,
(13) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = \gamma_1(-\frac{1}{2}y + xy) \otimes v_2^0 + (1-2x) \otimes v_1^1$,
(14) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_2^0 + 1 \otimes v_1^1$,
(15) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = \big(\gamma_1(-\frac{1}{2}y + xy)\big) \otimes v_1^0 + (1-2x) \otimes v_1^1$,
(16) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1$,
(17) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$.

Proposition A.8. *The supercomodule structures over $\mathcal{H}_2$ are defined in the different cases as follows.*

- *Case* $m_0 = 1,\ m_1 = 1.$

(1) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + (1-2x) \otimes v_1^1$,
(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^0 + 1 \otimes v_1^1$,
(3) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1$,
(4) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,
(5) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1$,
(6) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$.

- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_2 y \otimes v_1^0 + (1-2x) \otimes v_2^1$,
(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^0 + 1 \otimes v_1^1, \rho(v_2^1) = (\gamma_2 y - 2\gamma_2 xy) \otimes v_1^0 + 1 \otimes v_2^1$,
(3) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_2 x \otimes v_1^1 + 1 \otimes v_2^1$,
(4) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1 + \gamma_2 y \otimes v_2^1, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = (1-2x) \otimes v_2^1$,

(5) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^1 + (\gamma_2 y - 2\gamma_2 xy) \otimes v_2^1, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$,

(6) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = \gamma_1 y \otimes v_1^0 + \gamma_2 x \otimes v_1^1 + (1-2x) \otimes v_2^1$,

(7) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^0 + \gamma_2 x \otimes v_1^1 + 1 \otimes v_2^1$,

(8) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_2^1, \rho(v_1^1) = (1-2x) \otimes v_1^1 + \gamma_2 x \otimes v_2^1, \rho(v_2^1) = 1 \otimes v_2^1$,

(9) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_1 x \otimes v_1^1 + 1 \otimes v_2^1$,

(10) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1 + \gamma_1 x \otimes v_2^1, \rho(v_2^1) = (1-2x) \otimes v_2^1$,

(11) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = (1-2x) \otimes v_2^1$,

(12) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$.

- *Case* $m_0 = 2,\ m_1 = 1$.

(1) $\rho(v_1^0) = (1+\gamma_1 x) \otimes v_1^0 + \gamma_2 x \otimes v_2^0, \rho(v_2^0) = -\frac{2\gamma_1 + \gamma_1^2}{\gamma_2} x \otimes v_1^0 + (1 - (2+\gamma_1) x) \otimes v_2^0$,
$\rho(v_1^1) = (1-2x) \otimes v_1^1, (\gamma_2 \neq 0)$

(2) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_2^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1$,

(3) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 x \otimes v_2^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_2^0$,

(4) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0, \rho(v_1^1) = \gamma_2 y \otimes v_1^0 + (1-2x) \otimes v_1^1$,

(5) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_2^0 + (1-2x) \otimes v_1^1$,

(6) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + \gamma_2 y \otimes v_2^0 + (1-2x) \otimes v_1^1$,

(7) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_2^0) = (1-2x) \otimes v_2^0 + \gamma_2 y \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1$,

(8) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0 + (\gamma_2 y - 2\gamma_2 xy) \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,

(9) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^0 + (\gamma_2 y - 2\gamma_2 xy) \otimes v_2^0 + 1 \otimes v_1^1$,

(10) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1$,

(11) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^1, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1$,

(12) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0 + (\gamma_2 y - 2\gamma_2 xy) \otimes v_1^1 \rho(v_1^1) = 1 \otimes v_1^1$,

(13) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$,

(14) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + (1-2x) \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$,

(15) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_1^0 + 1 \otimes v_1^1$,

(16) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (\gamma_1 y - 2\gamma_1 xy) \otimes v_2^0 + 1 \otimes v_1^1$,

(17) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$,

(18) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1$.

Proposition A.9. *The supercomodule structures over $\mathcal{H}_4$ are defined in the different cases as follows.*

- *Case* $m_0 = 1,\ m_1 = 1$.

(1) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = (\gamma_1(y - 2xy)) \otimes v_1^0 + 1 \otimes v_1^1$,

(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + (1-2x) \otimes v_1^1,$
(3) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1,$
(4) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 (y-2xy)) \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1,$
(5) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1,$
(6) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1.$

- *Case* $m_0 = 1,\ m_1 = 2.$

(1) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1(y-2xy) \otimes v_1^1 + \gamma_2(y-2xy) \otimes v_2^1,$
$\rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = (1-2x) \otimes v_2^1,$
(2) $\rho(v_1^0) = (1-2x)\otimes v_1^0+\gamma_1 y\otimes v_1^1+\gamma_2 y\otimes v_2^1, \rho(v_1^1) = 1\otimes v_1^1, \rho(v_2^1) = 1\otimes v_2^1,$
(3) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = \gamma_1 y \otimes v_1^0 + (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_2 y \otimes v_1^0 + (1-2x) \otimes v_2^1,$
(4) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = \gamma_1(y-2xy) \otimes v_1^0 + 1 \otimes v_1^1, \rho(v_2^1) = \gamma_2(y-2xy) \otimes v_1^0 + 1 \otimes v_2^1,$
(5) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_1^1) = 1\otimes v_1^1, \rho(v_2^1) = \gamma_1 y\otimes v_1^0+\gamma_2 x\otimes v_1^1+(1-2x)\otimes v_2^1,$
(6) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_1(y-2xy) \otimes v_1^0 + \gamma_2 x \otimes v_1^1 + 1 \otimes v_2^1,$
(7) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1(y-2xy) \otimes v_1^1 \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = \gamma_2 x \otimes v_1^1 + 1 \otimes v_2^1,$
(8) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = \gamma_2 x \otimes v_1^1 + (1-2x) \otimes v_2^1,$
(9) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_1^1) = (1-2x) \otimes v_1^1, \rho(v_2^1) = (1-2x) \otimes v_2^1,$
(10) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1.$

- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 x \otimes v_2^0, \rho(v_2^0) = 1 \otimes v_2^0,$
$\rho(v_1^1) = \frac{\gamma_2}{\gamma_1}(-2y+4xy) \otimes v_1^0 + (\gamma_2(y-2xy)) \otimes v_2^0 + 1 \otimes v_1^1, (\gamma_1 \neq 0)$
(2) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1 x \otimes v_2^0, \rho(v_2^0) = (1-2x) \otimes v_2^0,$
$\rho(v_1^1) = \frac{2\gamma_2}{\gamma_1} y \otimes v_1^0 + \gamma_2 y \otimes v_2^0 + (1-2x) \otimes v_1^1, (\gamma_1 \neq 0)$
(3) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0,$
$\rho(v_1^1) = (\gamma_1(y-2xy)) \otimes v_1^0 + \gamma_2(y-2xy) \otimes v_2^0 + 1 \otimes v_1^1,$
(4) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = 1\otimes v_2^0, \rho(v_1^1) = \gamma_1 y\otimes v_1^0+\gamma_2 y\otimes v_2^0+(1-2x)\otimes v_1^1,$
(5) $\rho(v_1^0) = (1-2x)\otimes v_1^0+\gamma_1 y\otimes v_1^1, \rho(v_2^0) = (1-2x)\otimes v_2^0+\gamma_2 y\otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1,$
(6) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1(y-2xy)) \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0 + \gamma_2(y-2xy) \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1,$
(7) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = (1-2x)\otimes v_2^0, \rho(v_1^1) = \gamma_1(y-2xy)\otimes v_2^0+1\otimes v_1^1,$
(8) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + (1-2x) \otimes v_2^0, \rho(v_2^0) = \gamma_2 y \otimes v_1^0 + (1-2x) \otimes v_1^1,$
(9) $\rho(v_1^0) = 1\otimes v_1^0, \rho(v_2^0) = \gamma_1 x\otimes v_1^0+(1-2x)\otimes v_2^0+\gamma_2 y\otimes v_1^1, \rho(v_1^1) = 1\otimes v_1^1,$
(10) $\rho(v_1^0) = 1 \otimes v_1^0 + \gamma_1(y-2xy) \otimes v_1^1, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1,$
(11) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0 \rho(v_1^1) = \gamma_2(y-2xy) \otimes v_1^0 + 1 \otimes v_1^1,$
(12) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = \gamma_1 y \otimes v_2^0 + (1-2x) \otimes v_1^1,$
(13) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = \gamma_1 x \otimes v_1^0 + 1 \otimes v_2^0 + \gamma_2(y-2xy) \otimes v_1^1, \rho(v_1^1) = (1-2x) \otimes v_1^1,$
(14) $\rho(v_1^0) = (1-2x) \otimes v_1^0 + \gamma_1 y \otimes v_1^1, \rho(v_2^0) = \gamma_2 x \otimes v_1^0 + 1 \otimes v_2^0 - \frac{1}{2}\gamma_1\gamma_2 y \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1,$

(15) $\rho(v_1^0) = (1-2x) \otimes v_1^0, \rho(v_2^0) = (1-2x) \otimes v_2^0, \rho(v_1^1) = (1-2x) \otimes v_1^1$,
(16) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = 1 \otimes v_1^1$.

Proposition A.10. *The supercomodule structures over $\mathcal{H}_3$ are defined in the different cases as follows.*
• *Case* $m_0 = 1,\ m_1 = 0$. $\rho(v_1^0) = 1 \otimes v_1^0$.
• *Case* $m_0 = 0,\ m_1 = 1$. $\rho(v_1^1) = 1 \otimes v_1^1$.

Proposition A.11. *The supercomodule structures over $\mathcal{H}_3$ are defined in the different cases as follows.*
• *Case* $m_0 = 2,\ m_1 = 0$. $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0$,
• *Case* $m_0 = 0,\ m_1 = 2$. $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$.
• *Case* $m_0 = 1,\ m_1 = 1$.
(1) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (\gamma_1 x + \gamma_2 y) \otimes v_1^0 + 1 \otimes v_1^1$,
(2) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 x + \gamma_1 y) \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$.

Proposition A.12. *The supercomodule structures over $\mathcal{H}_3$ are defined in the different cases as follows.*
• *Case* $m_0 = 3,\ m_1 = 0$. $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_3^0) = 1 \otimes v_3^0$,
• *Case* $m_0 = 0,\ m_1 = 3$. $\rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1, \rho(v_3^1) = 1 \otimes v_3^1$.
• *Case* $m_0 = 1,\ m_1 = 2$.
(1) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_1^1) = (\gamma_1 x + \gamma_2 y) \otimes v_1^0 + 1 \otimes v_1^1, \rho(v_2^1) = (\gamma_3 x + \gamma_4 y) \otimes v_1^0 + 1 \otimes v_2^1$,
(2) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 x + \gamma_2 y) \otimes v_1^1 + (\gamma_3 x + \gamma_4 y) \otimes v_2^1, \rho(v_1^1) = 1 \otimes v_1^1, \rho(v_2^1) = 1 \otimes v_2^1$.
• *Case* $m_0 = 2,\ m_1 = 1$.
(1) $\rho(v_1^0) = 1 \otimes v_1^0 + (\gamma_1 x + \gamma_2 y) \otimes v_1^1, \rho(v_2^0) = 1 \otimes v_2^0 + (\gamma_3 x + \gamma_4 y) \otimes v_1^1, \rho(v_1^1) = 1 \otimes v_1^1$,
(2) $\rho(v_1^0) = 1 \otimes v_1^0, \rho(v_2^0) = 1 \otimes v_2^0, \rho(v_1^1) = (\gamma_1 x + \gamma_2 y) \otimes v_1^0 + (\gamma_3 x + \gamma_4 y) \otimes v_2^0 + 1 \otimes v_1^1$.

A.3. **Low dimensional supermodules superalgebras over $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$.** Let A be the superalgebra of dimension $m = m_0 + m_1$, where m_0 the even part dimension and m_1 is the odd part dimension. Let $\{v_p^q\}$ be a basis of A, where $q \in \{0,1\}, p \in \{1, \cdots, m_q\}$.

Proposition A.13. *The supermodule superalgebra structures over $\mathcal{H}_1$ and $\mathcal{H}_4$ are defined in the different cases as follows.*
• *Case* $m_0 = 1,\ m_1 = 0$. $x \rhd v_1^0 = 0$.
• *Case* $m_0 = 2,\ m_1 = 0$.
(1) $v_2^0 v_2^0 = 0,\ x \rhd v_2^0 = v_2^0, x \rhd v_1^0 = 0$, (2) $v_2^0 v_2^0 = v_2^0,\ x \rhd v_2^0 = \frac{-1}{2} v_1^0 + v_2^0, x \rhd v_1^0 = 0$.
• *Case* $m_0 = 1,\ m_1 = 1$. $v_1^1 v_1^1 = 0$,
(1) $y \rhd v_1^1 = \lambda_1 v_1^0, x \rhd v_1^1 = xy \rhd v_1^1 = 0$, (2) $x \rhd v_1^1 = v_1^1, y \rhd v_1^1 = xy \rhd v_1^1 = 0$.

Proposition A.14. *The supermodule superalgebra structures over $\mathcal{H}_1$ and $\mathcal{H}_4$ are defined in the different cases as follows.*
• *Case* $m_0 = 3,\ m_1 = 0$.
(1) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0 v_3^0 = v_3^0,\ x \rhd v_2^0 = x \rhd v_3^0 = \frac{1}{2}(-v_1^0 + v_2^0 + v_3^0), x \rhd v_1^0 = 0$,
(2) $v_2^0 v_2^0 = v_2^0, v_2^0 v_3^0 = v_3^0 v_2^0 = v_3^0, v_3^0 v_3^0 = 0,\ x \rhd v_3^0 = v_3^0, x \rhd v_1^0 = x \rhd v_2^0 = 0$,

(3) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_3^0 = v_3^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0,$
(4) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \lambda_1 v_2^0, x \triangleright v_1^0 = 0,$
(5) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = v_3^0, x \triangleright v_1^0 = 0,$
(6) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0, v_3^0v_2^0 = v_3^0v_3^0 = 0, x\triangleright v_2^0 = \lambda_1 v_3^0, x\triangleright v_3^0 = v_3^0, x\triangleright v_1^0 = 0.$
• *Case* $m_0 = 1,\ m_1 = 2.$ $v_1^1v_1^1 = v_1^1v_2^1 = v_2^1v_1^1 = v_2^1v_2^1 = 0,$
(1) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \lambda_1 v_1^1, y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$
• *Case* $m_0 = 2,\ m_1 = 1.$
(1) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_1^1 = v_1^1, y \triangleright v_2^0 = xy \triangleright v_2^0 = (1-a)\lambda_1 v_1^1,$
$y \triangleright v_1^1 = xy \triangleright v_1^1 = a\lambda_2 v_2^0,$
(2) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0,\ y \triangleright v_2^0 = (1-a)\lambda_1 v_1^1, y \triangleright v_1^1 = a\lambda_2 v_2^0, x \triangleright v_2^0 = x \triangleright v_1^1 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(3) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0,\ x\triangleright v_2^0 = (1-a)v_2^0, x\triangleright v_1^1 = av_1^1, y\triangleright v_2^0 = y \triangleright v_1^1 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(4) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = 0, v_1^1v_1^1 = v_2^0,\ x \triangleright v_1^1 = (1-a)v_1^1, y \triangleright v_1^1 = a\lambda_1 v_2^0, x \triangleright v_2^0 = y \triangleright v_2^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(5) $v_2^0v_2^0 = v_1^1v_1^1 = v_2^0, v_0^2v_1^1 = v_1^1v_2^0 = v_1^1,\ x \triangleright v_1^1 = (1-a)v_1^1, y \triangleright v_1^1 = a\lambda_1 v_2^0, x \triangleright v_2^0 = y \triangleright v_2^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(6) $v_2^0v_2^0 = v_2^0, v_2^0v_1^1 = v_1^1, v_1^1v_1^1 = v_1^1v_2^0 = 0,\ x \triangleright v_1^1 = (1-a)v_1^1, y \triangleright v_2^0 = a\lambda_1 v_1^1, x \triangleright v_2^0 = y \triangleright v_1^1 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$
(7) $v_2^0v_2^0 = v_2^0, v_2^0v_1^1 = v_1^1v_2^0 = v_1^1, v_1^1v_1^1 = 0,\ x \triangleright v_1^1 = (1-a)v_1^1, y \triangleright v_1^1 = a\lambda_1 v_2^0, x \triangleright v_2^0 = y \triangleright v_2^0 = xy \triangleright v_2^0 = xy \triangleright v_1^1 = 0,$ *with* $a = \{0, 1\}.$

Proposition A.15. *The supermodule superalgebra structures over* $\mathcal{H}_2$ *are defined in the different cases as follows.*
• *Case* $m_0 = 1,\ m_1 = 0.$ $x \triangleright v_1^0 = 0,$
• *Case* $m_0 = 2,\ m_1 = 0.$
(1) $v_2^0v_2^0 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_1^0 = 0,$ (2) $v_2^0v_2^0 = v_2^0,\ x \triangleright v_2^0 = \frac{-1}{2}v_1^0 + v_2^0, x \triangleright v_1^0 = 0.$
• *Case* $m_0 = 1,\ m_1 = 1.$ $v_1^1v_1^1 = 0,\ x \triangleright v_1^1 = v_1^1, y \triangleright v_1^1 = \lambda_1 v_1^0, xy \triangleright v_1^1 = 0.$

Proposition A.16. *The supermodule superalgebra structures over* $\mathcal{H}_2$ *are defined in the different cases as follows.*
• *Case* $m_0 = 3,\ m_1 = 0.$
(1) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = v_3^0,\ x \triangleright v_2^0 = x \triangleright v_3^0 = \frac{1}{2}(-v_1^0 + v_2^0 + v_3^0), x \triangleright v_1^0 = 0,$
(2) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0v_2^0 = v_3^0, v_3^0v_3^0 = 0,\ x \triangleright v_3^0 = v_3^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0,$
(3) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_3^0 = v_3^0, x \triangleright v_1^0 = x \triangleright v_2^0 = 0,$
(4) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = \lambda_1 v_2^0, x \triangleright v_1^0 = 0,$
(5) $v_2^0v_2^0 = v_2^0v_3^0 = v_3^0v_2^0 = v_3^0v_3^0 = 0,\ x \triangleright v_2^0 = v_2^0, x \triangleright v_3^0 = v_3^0, x \triangleright v_1^0 = 0,$
(6) $v_2^0v_2^0 = v_2^0, v_2^0v_3^0 = v_3^0, v_3^0v_2^0 = v_3^0v_3^0 = 0, x\triangleright v_2^0 = \lambda_1 v_3^0, x\triangleright v_3^0 = v_3^0, x\triangleright v_1^0 = 0.$
• *Case* $m_0 = 1,\ m_1 = 2.$ $v_1^1v_1^1 = v_1^1v_2^1 = v_2^1v_1^1 = v_2^1v_2^1 = 0,$
(1) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = v_2^1, y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0,$
(2) $x \triangleright v_1^1 = v_1^1, x \triangleright v_2^1 = \lambda_1 v_1^1, y \triangleright v_1^1 = y \triangleright v_2^1 = xy \triangleright v_1^1 = xy \triangleright v_2^1 = 0.$
• *Case* $m_0 = 2,\ m_1 = 1.$
(1) $v_2^0v_2^0 = v_2^0v_1^1 = v_1^1v_2^0 = v_1^1v_1^1 = 0,\ x\triangleright v_2^0 = av_2^0, x\triangleright v_1^1 = (1-a)v_1^1, y\triangleright v_2^0 =$

$xy \rhd v_2^0 = (1-a)\lambda_1 v_1^1, y \rhd v_1^1 = xy \rhd v_1^1 = a\lambda_2 v_2^0,$
(2) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1 v_1^1 = 0,$ $x \rhd v_2^0 = (1-a)v_2^0, x \rhd v_1^1 = av_1^1, y \rhd v_2^0 = (1-a)\lambda_1 v_1^1, y \rhd v_1^1 = a\lambda_2 v_2^0, xy \rhd v_2^0 = xy \rhd v_1^1 = 0,$
(3) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1 v_1^1 = 0,$ $x \rhd v_2^0 = v_2^0, x \rhd v_1^1 = v_1^1, y \rhd v_2^0 = y \rhd v_1^1 = xy \rhd v_2^0 = xy \rhd v_1^1 = 0,$
(4) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = 0, v_1^1 v_1^1 = v_2^0,$ $x \rhd v_1^1 = v_1^1, y \rhd v_1^1 = \lambda_1 v_2^0, x \rhd v_2^0 = y \rhd v_2^0 = xy \rhd v_2^0 = xy \rhd v_1^1 = 0,$
(5) $v_2^0 v_2^0 = v_1^1 v_1^1 = v_2^0, v_0^2 v_1^1 = v_1^1 v_2^0 = v_1^1,$ $x \rhd v_1^1 = v_1^1, y \rhd v_1^1 = \lambda_1 v_2^0, x \rhd v_2^0 = y \rhd v_2^0 = xy \rhd v_2^0 = xy \rhd v_1^1 = 0,$
(6) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1, v_1^1 v_1^1 = v_1^1 v_2^0 = 0,$ $x \rhd v_1^1 = v_1^1, y \rhd v_2^0 = xy \rhd v_2^0 = \lambda_1 v_1^1, x \rhd v_2^0 = y \rhd v_1^1 = xy \rhd v_1^1 = 0,$
(7) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1, v_1^1 v_1^1 = 0,$ $x \rhd v_1^1 = v_1^1, y \rhd v_1^1 = \lambda_1 v_2^0, x \rhd v_2^0 = y \rhd v_2^0 = xy \rhd v_2^0 = xy \rhd v_1^1 = 0,$ *with* $a = \{0, 1\}.$

Proposition A.17. *The supermodule superalgebra structures over* $\mathcal{H}_3$ *are defined in the different cases as follows.*
- *Case* $m_0 = 1,\ m_1 = 0.$ $xy \rhd v_1^0 = 0,$
- *Case* $m_0 = 2,\ m_1 = 0.$ $xy \rhd v_1^0 = xy \rhd v_2^0 = 0,$
- *Case* $m_0 = 1,\ m_1 = 1.$ $v_1^1 v_1^1 = 0, x \rhd v_1^1 = \lambda_1 v_1^0, y \rhd v_1^1 = \lambda_2 v_1^0, xy \rhd v_1^1 = 0.$

Proposition A.18. *The supermodule superalgebra structures over* $\mathcal{H}_3$ *are defined in the different cases as follows.*
- *Case* $m_0 = 3,\ m_1 = 0.$ $xy \rhd v_1^0 = xy \rhd v_2^0 = xy \rhd v_3^0 = 0,$
- *Case* $m_0 = 1,\ m_1 = 2.$ $x \rhd v_1^1 = x \rhd v_2^1 = y \rhd v_1^1 = y \rhd v_2^1 = xy \rhd v_1^1 = xy \rhd v_2^1 = 0,$
- *Case* $m_0 = 2,\ m_1 = 1.$

(1) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1 v_1^1 = 0,$ $xy \rhd v_2^0 = xy \rhd v_1^1 = 0, x \rhd v_2^0 = (1-a)\lambda_1 v_1^1, x \rhd v_1^1 = a\lambda_2 v_2^0, y \rhd v_2^0 = (1-a)\lambda_3 v_1^1, y \rhd v_1^1 = a\lambda_4 v_2^0.$
(2) $v_2^0 v_2^0 = v_2^0 v_1^1 = v_1^1 v_2^0 = 0, v_1^1 v_1^1 = v_2^0,$ $x \rhd v_1^1 = \lambda_1 v_2^0, y \rhd v_1^1 = \lambda_2 v_2^0, xy \rhd v_2^0 = xy \rhd v_1^1 = x \rhd v_2^0 = y \rhd v_2^0 = 0,$
(3) $v_2^0 v_2^0 = v_1^1 v_1^1 = v_2^0, v_0^2 v_1^1 = v_1^1 v_2^0 = v_1^1,$ $x \rhd v_1^1 = \lambda_1 v_2^0, y \rhd v_1^1 = \lambda_2 v_2^0, xy \rhd v_2^0 = xy \rhd v_1^1 = x \rhd v_2^0 = y \rhd v_2^0 = 0,$
(4) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1, v_1^1 v_1^1 = v_1^1 v_2^0 = 0,$ $x \rhd v_2^0 = \lambda_1 v_1^1, y \rhd v_2^0 = \lambda_2 v_1^1, xy \rhd v_2^0 = xy \rhd v_1^1 = x \rhd v_1^1 = y \rhd v_1^1 = 0,$
(5) $v_2^0 v_2^0 = v_2^0, v_2^0 v_1^1 = v_1^1 v_2^0 = v_1^1, v_1^1 v_1^1 = 0,$ $x \rhd v_1^1 = \lambda_1 v_2^0, y \rhd v_1^1 = \lambda_2 v_2^0, xy \rhd v_2^0 = xy \rhd v_1^1 = x \rhd v_2^0 = y \rhd v_2^0 = 0.$

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